Convergence Analysis of the Fast Subspace Descent (FASD) Method for Convex Optimization Problems – Especially Those of Cahn-Hilliard Type –

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A long-form pre-print of this work, with loads of extra details, can be found on my website, publication number 77.

Some Ultimate and Long-Term Objectives

1. Create and demonstrate a provably globally convergent nonlinear (FAS) multigrid algorithm for the (Steady) Cahn-Hilliard.
2. The algorithm should have optimal or near optimal complexity, i.e., it should be fast.
3. The algorithm should work and work efficiently for a broad range of nonlinearities in the equation and certain types of degenerate or nearly degenerate mobilities.

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We don't get everything, but we get a lot...

Outline

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Recall the Cahn-Hilliard Equation

The Cahn-Hilliard equation in mixed formulation (Cahn, 1961):

$$\begin{aligned} \partial_t\phi &= \varepsilon\Delta\mu && \text{in} \quad \Omega, \\ \mu &= \varepsilon^{-1}\phi^3 - \varepsilon^{-1}\phi - \varepsilon\Delta\phi && \text{in} \quad \Omega, \\ \partial_n\phi = \partial_n\mu &= 0 && \text{on} \quad \partial\Omega, \end{aligned}$$

with $\phi(0) = \phi_0$, where $\varepsilon > 0$ is the interfacial parameter.

Mixed weak formulation: find $\phi \in L^\infty\left(0, T; H^1(\Omega)\right) \cap C\left([0, T], L^2(\Omega)\right)$, $\partial_t\phi \in L^2\left(0, T; H^{-1}(\Omega)\right)$ and $\mu \in L^2\left(0, T; H^1(\Omega)\right)$ such that

$$\begin{aligned} \langle\partial_t\phi, \chi\rangle + \varepsilon\left(\nabla\mu, \nabla\chi\right) &= 0 && \forall\chi \in H^1(\Omega), \\ \varepsilon^{-1}\left(\phi^3 - \phi, \varphi\right) + \varepsilon\left(\nabla\phi, \nabla\varphi\right) - \left(\mu, \varphi\right) &= 0 && \forall\varphi \in H^1(\Omega), \end{aligned}$$

for almost all $t \in (0, T)$, with $\phi(0) = \phi_0$. Note that BCs are natural.

Conserved Gradient Flow

Consider the typical Cahn-Hilliard free energy (Cahn and Hilliard, 1957)

$$\mathcal{E}(\phi) = \int_\Omega \left\{ \frac{1}{4\varepsilon}\phi^4 - \frac{1}{2\varepsilon}\phi^2 + \frac{\varepsilon}{2}|\nabla\phi|^2 \right\} d\mathbf{x}.$$

The chemical potential is

$$\mu = \delta_\phi \mathcal{E} = \varepsilon^{-1}\phi^3 - \varepsilon^{-1}\phi - \varepsilon\Delta\phi.$$

Weak solutions dissipate the energy at the rate

$$\mathcal{E}(\phi(s)) + \int_0^s \|\nabla\mu\|_{L^2}^2 \, dt = \mathcal{E}(\phi(0)), \quad \left(d_t\mathcal{E}(\phi) = -\|\nabla\mu\|_{L^2}^2 \right).$$

Mass conservation:

$$\int_\Omega (\phi(\mathbf{x},t) - \phi(\mathbf{x},0))\, d\mathbf{x} = 0, \text{ a.e. } t > 0, \quad \left(d_t \int_\Omega \phi(\mathbf{x},t)\, d\mathbf{x} = 0 \right).$$

Time Discretization

Let $0 = t_0 \leq t_1 \leq \cdots t_M = T$, be a uniform partition of $[0, T]$, with $\tau = t_m - t_{m-1}$.

Given $\phi^{m-1} \in H^1(\Omega)$, find ϕ^m, $\mu^m \in H^1(\Omega)$ such that

$$\left(\delta_\tau \phi^m, \chi\right) + \varepsilon\left(\nabla \mu^m, \nabla \chi\right) = 0 \qquad \forall \chi \in H^1(\Omega),$$

$$\frac{1}{\varepsilon}\left(\left(\phi^m\right)^3 - \phi^{m^\star}, \psi\right) + \varepsilon\left(\nabla \phi^m, \nabla \psi\right) - \left(\mu^m, \psi\right) = 0 \qquad \forall \psi \in H^1(\Omega),$$

where

$$\delta_\tau \phi^m = \frac{\phi^m - \phi^{m-1}}{\tau},$$

and

$$m^\star = \begin{cases} m & \text{for} \quad \text{Backward Euler} \\ & \text{or} \\ m-1 & \text{for} \quad \text{Convex Splitting} \end{cases}.$$

Mixed, Fully Discrete Scheme

For any $1 \leq m \leq M$, given $\phi_h^{m-1} \in S_h$ find $\phi_h^m, \mu_h^m \in S_h$ such that

$$
\begin{aligned}
(\delta_\tau \phi_h^m, \chi) + (\nabla \mu_h^m, \nabla \chi) &= 0, \qquad \forall \chi \in S_h, \\
\varepsilon^{-1}\left((\phi_h^m)^3 - \phi_h^{m^\star}, \psi \right) + \varepsilon \left(\nabla \phi_h^m, \nabla \psi\right) - (\mu_h^m, \psi) &= 0, \qquad \forall \psi \in S_h,
\end{aligned}
$$

where

$$
S_h := \left\{ v \in C^0\left(\overline{\Omega}\right) \;\middle|\; v|_K \in P_1(K),\ K \in \mathcal{T}_h \right\} \subset H^1(\Omega),
$$

and

$$
\phi_h^0 := R_h \phi_0.
$$

$R_h : H^1(\Omega) \to S_h$ is the elliptic (Ritz) projection.

It is easy to see that the scheme is discretely mass conservative:

$$
\left(\phi_h^m - \bar{\phi}_0, 1\right) = 0, \quad \forall\ m \geq 1.
$$

A Stationary Nonlinear Problem

$$\begin{aligned} u - \Delta w &= f, && \text{in } \Omega, \\ |u|^{p-2}u - \Delta u - w &= g, && \text{in } \Omega, \\ \partial_n u = \partial_n w &= 0, && \text{on } \partial\Omega, \end{aligned}$$

where $2 \leq p < \infty$, $f \in H^1(\Omega) \cap L^p(\Omega)$, and $g \in L^r(\Omega)$ with $\frac{1}{p} + \frac{1}{r} = 1$. A mixed weak formulation is written as follows: find $u \in H^1(\Omega) \cap L^p(\Omega)$ and $w \in H^1(\Omega)$ such that

$$\begin{aligned} (u, \chi) + (\nabla w, \nabla \chi) &= (f, \chi) && \forall \chi \in H^1(\Omega), \\ \left(|u|^{p-2}u, \psi\right) + (\nabla u, \nabla \psi) - (w, \psi) &= (g, \psi) && \forall \psi \in H^1(\Omega) \cap L^p(\Omega). \end{aligned}$$

Mass is conserved in the sense that $(u - f, 1) = 0$.

An Equivalent Convex Optimization Problem

Equivalently, find $v \in \mathring{H}^1(\Omega) \cap L^p(\Omega)$ satisfying $E(v) = \inf_{\tilde{v} \in \mathcal{A}} E(\tilde{v})$ given the energy and the admissible set

$$E(v) := \frac{1}{2} \left\| v - f + \bar{f} \right\|_{-1}^2 + \frac{1}{2} \left\| \nabla v \right\|_{L^2}^2 + \frac{1}{p} \left\| v + \bar{f} \right\|_{L^p}^p - (g, v),$$

$$\mathcal{A} := \mathring{H}^1(\Omega) \cap L^p(\Omega).$$

It is straightforward to show E has a unique global minimizer, and the associated Euler-Lagrange equation is

$$\left(|v + \bar{f}|^{p-2}(v + \bar{f}), \psi \right) + (\nabla v, \nabla \psi) + \left(\mathrm{T}(v - f + \bar{f}), \psi \right) = (g, \psi), \quad \forall \psi \in \mathcal{A}.$$

$T = A^{-1} = (-\Delta)^{-1}$. The chemical potential equation can be recovered via $-w_\star = \mathrm{T}(v - f + \bar{f})$ to get

$$(v, \chi) + (\nabla w_\star, \nabla \chi) = (f, \chi), \quad \forall \chi \in \mathring{H}^1(\Omega).$$

Outline

A Drop in a Shear Flow

Isothermal Buoyancy with Mixing

Convection Flows

Lava Lamps

Tumor Growth

A Block Copolymer Melt in a Shear Flow

Outline

Two Classes of Nonlinear Stationary PDE

- A Second-Order Problem: Find $u : \Omega \to \mathbb{R}$ such that

$$-\varepsilon \Delta u + |u|^{p-2} u = f \quad \text{in} \quad \Omega,$$

  with $u = 0$ on $\partial\Omega$ and $p \geq 2$.

- A Fourth-Order Problem: Find $u, w : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} u - \Delta w &= g \quad \text{in} \quad \Omega \\ -\varepsilon \Delta u + |u|^{p-2} u - w &= f \quad \text{in} \quad \Omega, \end{aligned}$$

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These problems are the Euler equations of certain convex energies.

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These problems are the Euler equations of certain convex energies.

Now, let me tell you a little lie: The first and second problems are morally the same. An algorithm that works for the first will work for the second.

A General Convex Optimization Problem

Find $u \in \mathcal{V}$, where $\mathcal{V}$ is a Hilbert space, such that

$$u = \operatorname*{argmin}_{w \in \mathcal{V}} E(w).$$

- How does one compute solutions or approximate solutions?
- For a good approximation, one must usually solve

$$\mathcal{N}(\mathbf{u}) = \mathbf{f},$$

  N nonlinear equations in N unknowns, where N is very large.
- Can approximate solutions be computed efficiently via iteration?
- Will the convergence rate of our iterative method depend upon N?
- Convergence in $O(N)$ or $O(N \log(N))$ operations?

Assumptions on the Energy

We assume that the energy functional $E(\,\cdot\,) : \mathcal{V} \to \mathbb{R}$ is Fréchet differentiable for all points $v \in \mathcal{V}$.

Energy Assumptions

(E1) (Strong convexity/ellipticity): There is a constant $\mu > 0$ such that

$$\mu \, \|w - v\|_{\mathcal{V}}^2 \leq \langle E'(w) - E'(v), w - v \rangle, \tag{3}$$

for all $v, w \in \mathcal{V}$, where $\langle \,\cdot\,, \,\cdot\, \rangle$ is the dual pairing between $\mathcal{V}'$ and $\mathcal{V}$.

(E2) (Lipschitz continuity of derivatives): For fixed $u_0 \in \mathcal{V}$, there exists a constant L such that, for all $v, w \in \mathcal{B} := \{v \in \mathcal{V} \mid E(v) \leq E(u_0)\}$,

$$\|E'(w) - E'(v)\|_{\mathcal{V}'} \leq L \|w - v\|_{\mathcal{V}}. \tag{4}$$

Theorem (Existence and Uniqueness of Minimizers)

If E satisfies assumption (E1), then, for all $w, v \in \mathcal{V}$

$$E(w) - E(v) \geq \langle E'(v), w - v \rangle + \frac{\mu}{2} \|w - v\|_{\mathcal{V}}^2 . \tag{5}$$

Consequently, E is strictly convex and coercive. Furthermore, there is a unique element $u \in \mathcal{V}$ with the property that

$$E(u) \leq E(v), \quad \forall\, v \in \mathcal{V}, \quad \textit{and} \quad E(u) < E(v), \quad \forall\, v \neq u,$$

and this global minimizer satisfies Euler's equation

$$\langle E'(u), w \rangle = 0, \quad \forall\; w \in \mathcal{V}. \tag{6}$$

Remark

It is (4) that we want to solve; this is typically a nonlinear PDE or integral equation, et cetera.

Further Properties

Lemma (Upper and Lower Lipschitz)

Suppose E satisfies assumptions (E1) and (E2). For all $v, w \in \mathcal{B}$,

$$\mu \left\| w - v \right\|_{\mathcal{V}}^2 \leq \langle E'(w) - E'(v), w - v \rangle \leq L \left\| w - v \right\|_{\mathcal{V}}^2 .$$

Furthermore the lower bound holds for all $v, w \in \mathcal{V}$.

Proposition ($\mathcal{B}$ is Convex)

If E satisfies (E1), the bounded energy set,

$$\mathcal{B} := \{ v \in \mathcal{V} \mid E(v) \leq E(u_0) \} ,$$

is convex.

Lemma (Quadratic Energy Trap)

Suppose E satisfies assumptions (E1) and (E2). For all $v, w \in \mathcal{B}$,

$$\frac{\mu}{2}\|w-v\|_{\mathcal{V}}^2 + \langle E'(v), w-v\rangle \le E(w) - E(v) \le \langle E'(v), w-v\rangle + \frac{L}{2}\|w-v\|_{\mathcal{V}}^2 .$$

Furthermore the lower bound holds for all $v, w \in \mathcal{V}$. In addition, suppose $u \in \mathcal{B}$ is the minimizer of E, then for all $w \in \mathcal{B}$,

$$\frac{\mu}{2}\|w-u\|_{\mathcal{V}}^2 \le E(w) - E(u) \le \frac{L}{2}\|w-u\|_{\mathcal{V}}^2 . \quad \textit{(Energy Trap)}$$

Again the lower bound holds for all $w \in \mathcal{V}$.

Proof.

Use Taylor's Theorem with integral remainder, using that $\mathcal{B}$ is convex for the upper bound. □

Lemma (The Reciprocal Upper Bound)

Suppose that E satisfies Assumption (E1) and $u \in \mathcal{V}$ is the minimizer of E; then for all $v \in \mathcal{V}$,

$$0 \leq E(v) - E(u) \leq \frac{1}{2\mu} \| E'(v) \|_{\mathcal{V}'}^2 . \tag{7}$$

Proof.

This follows by Taylor's Theorem with integral remainder, and the Riesz Representation Theorem. □

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This will be a key estimate in the convergence analysis.

Lemma (Quadratic Energy Traps for Energy Sections)

Suppose that E satisfies (E1) – (E2), $\xi \in \mathcal{B}$ is arbitrary, and $\mathcal{W} \subseteq \mathcal{V}$ is a subspace. Define the energy section

$$J(w) := E(\xi + w), \quad \forall\, w \in \mathcal{W}.$$

Then $J : \mathcal{W} \to \mathbb{R}$ is differentiable, strongly convex, and there exists a unique element $\eta \in \mathcal{W}$ such that $\xi + \eta \in \mathcal{B}$, η is the unique global minimizer of J, and

$$\langle E'(\xi + \eta), w\rangle = \langle J'(\eta), w\rangle = 0, \quad \forall\, w \in \mathcal{W}.$$

Furthermore, for all $w \in \mathcal{W}$ with $w + \xi \in \mathcal{B}$,

$$\frac{\mu}{2} \|w - \eta\|_{\mathcal{V}}^2 \le J(w) - J(\eta) = E(\xi + w) - E(\xi + \eta) \le \frac{L}{2} \|w - \eta\|_{\mathcal{V}}^2 .$$

The lower bound holds for any $w \in \mathcal{W}$, without restriction.

Subspace Decompositions

Suppose that

$$\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 + \cdots + \mathcal{V}_N, \qquad \mathcal{V}_i \subseteq \mathcal{V}, \quad i = 1, \ldots, N.$$

Assumptions on Subspace Decompositions

(SS1) Stability: There is a constant $C_A > 0$, such that, for every $v \in \mathcal{V}$, there exists $v_i \in \mathcal{V}_i$, $i = 1, \cdots, N$, with the property that

$$v = \sum_{i=1}^{N} v_i, \quad \text{and} \quad \sum_{i=1}^{N} \|v_i\|_{\mathcal{V}}^2 \le C_A^2 \|v\|_{\mathcal{V}}^2.$$

(SS2) Strengthened CS: There is a constant $C_S > 0$, such that, for any $w_{i,j} \in \mathcal{B}$, $u_i \in \mathcal{V}_i$, $v_i \in \mathcal{V}_i$, with $w_{i,j} + u_i \in \mathcal{B}$,

$$\sum_{i=1}^{N} \sum_{j=i+1}^{N} \langle E'(w_{i,j} + u_j) - E'(w_{i,j}), v_i \rangle \le C_S \sqrt{\sum_{i=1}^{N} \|u_i\|_{\mathcal{V}}^2} \sqrt{\sum_{i=1}^{N} \|v_i\|_{\mathcal{V}}^2}.$$

(Xu, SIREV 1992) and (Tai and Xu, Math Comp, 2001)

Outline

Successive Subspace Optimization (SSO) Algorithm

Result: $u^{k+1} = \mathrm{SSO}(u^k)$

$v_0 = u^k$;

for $i = 1 : N$ **do**

Define an energy section along $\mathcal{V}_i$:

$$J_i(w) := E(v_{i-1} + w), \quad \forall w \in \mathcal{V}_i;$$

Compute the subspace correction:

$$e_i = \operatorname*{argmin}_{w \in \mathcal{V}_i} J_i(w); \tag{8}$$

Apply the subspace correction:

$$v_i = v_{i-1} + e_i;$$

end

$u^{k+1} = v_N$;

Tai and Xu, Math. Comp. (2001).

Fundamental Orthogonality and Gauss-Seidel

Remark

The "correction" e_i computed in SSO is uniquely defined since J_i inherits the convexity of E. We have the orthogonality condition

$$\langle E'(v_i), w\rangle = \langle E'(v_{i-1} + e_i), w\rangle = \langle J'(e_i), w\rangle = 0, \quad \forall\, w \in \mathcal{V}_i.$$

The condition

$$\langle E'(v_i), w\rangle = 0, \quad \forall\, w \in \mathcal{V}_i,$$

is referred to as the fundamental orthogonality (FO) of the solver.

Remark

SSO method can be considered as a generalization of the **nonlinear Gauss-Seidel** *methodology.*

Remark

Of course, we always decrease the energy in SSO:

$$E(u^k) = E(v_0) \geq E(v_1) \geq \cdots \geq E(v_N) = E(u^{k+1}).$$

Theorem (The Golden Key)

Suppose that $\{d_k\}_{k=0}^{\infty}$, $\{\delta_k\}_{k=0}^{\infty}$, $\{\alpha_k\}_{k=0}^{\infty}$ *are sequences of non-negative real numbers, the first two having the relationship*

$$\delta_k = d_k - d_{k+1}, \quad k = 0, 1, 2, \cdots .$$

Assume that there are constants $C_L, C_U > 0$*, independent of* k*, such that*

$$C_L \alpha_k \le \delta_k \quad \text{and} \quad d_{k+1} \le C_U \alpha_k .$$

Then

$$d_{k+1} \le \frac{C_U}{C_L + C_U} d_k, \quad k = 0, 1, 2, \cdots . \tag{9}$$

Consequently $\{d_k\}$ *converges monotonically, and (at least) linearly to 0.*

Proof.

$$d_{k+1} \le C_U \alpha_k = \frac{C_U}{C_L} C_L \alpha_k \le \frac{C_U}{C_L} \delta_k = \frac{C_U}{C_L}(d_k - d_{k+1}).$$

□

Corollary (Golden Key Strategy: Lower and Upper Energy Bounds)

Suppose that there exist positive constants C_L and C_U such that

$$E(u^k) - E(u^{k+1}) =: \delta_k \geq C_L \alpha_k = C_L \sum_{i=1}^{N} \|e_i\|_{\mathcal{V}}^2, \tag{10}$$

$$E(u^{k+1}) - E(u) =: d_{k+1} \leq C_U \alpha_k = C_U \sum_{i=1}^{N} \|e_i\|_{\mathcal{V}}^2. \tag{11}$$

Then

$$E(u^{k+1}) - E(u) \leq \rho \left(E(u^k) - E(u) \right), \quad \rho := \frac{C_U}{C_L + C_U},$$

and $E(u^k)$ converges monotonically, and (at least) linearly to $E(u)$, at the linear rate ρ. Furthermore, u^k converges at least linearly to u.

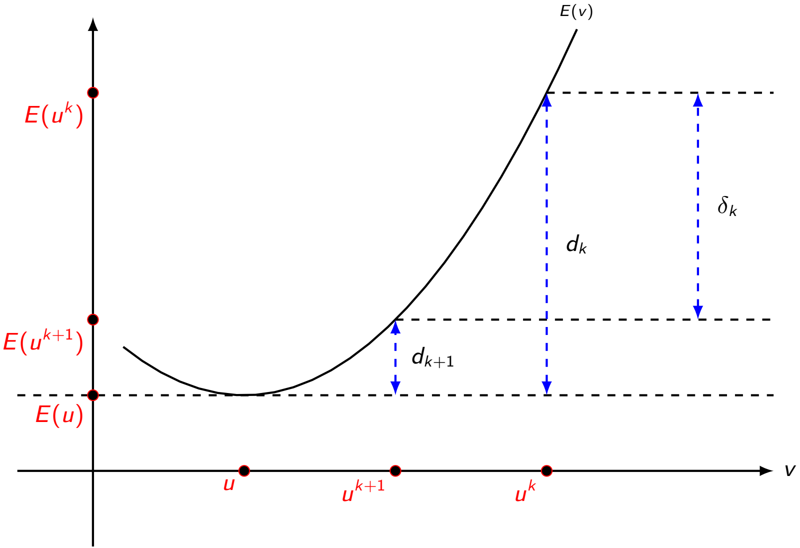

Figure: The sequences $\{d_k\}$ and $\{\delta_k\}$.

Theorem (SSO Lower Bound)

Let u^k be the k-th iteration and $u^{k+1} = \mathrm{SSO}(u^k)$. If E is strongly convex in the sense of satisfying (E1), then

$$\delta_k = E(u^k) - E(u^{k+1}) \geq C_L \sum_{i=1}^{N} \|e_i\|_{\mathcal{V}}^2, \quad C_L := \frac{\mu}{2}.$$

Proof.

Since the fundamental orthogonality, $J_i'(e_i) = E'(v_i) = 0$ in $\mathcal{V}_i'$, holds, and $e_i = v_i - v_{i-1} \in \mathcal{V}_i$, in view of the quadric energy traps for J_i, we have

$$E(v_{i-1}) - E(v_i) = J_i(0) - J_i(e_i) \geq \frac{\mu}{2} \|e_i\|_{\mathcal{V}}^2 . \quad \text{(FO + Lower Trap)}$$

which implies

$$E(u^k) - E(u^{k+1}) = \sum_{i=1}^{N} (E(v_{i-1}) - E(v_i)) \geq \frac{\mu}{2} \sum_{i=1}^{N} \|e_i\|_{\mathcal{V}}^2. \quad \text{(Telescope)}$$

□

Theorem (SSO Upper Bound)

Let u^{k+1} be the $k+1^{\mathrm{st}}$ iteration in the SSO algorithm. Suppose that the space decomposition satisfies Assumptions (SS1) and (SS2) and the energy E satisfies Assumption (E1), then we have

$$d_{k+1} = E(u^{k+1}) - E(u) \leq C_U \sum_{i=1}^{N} \|e_i\|_{\mathcal{V}}^2, \quad C_U := \frac{C_S^2 C_A^2}{2\mu}.$$

Proof (1 of 3)

Using the Reciprocal Upper Bound Lemma, with the choice $v = u^{k+1}$ in (7), we have

$$d_{k+1} = E(u^{k+1}) - E(u) \leq \frac{1}{2\mu} \|E'(u^{k+1})\|_{\mathcal{V}'}^2.$$

Let's estimate the operator norm.

Proof (2 of 3)

For any $w \in \mathcal{V}$, we choose a stable decomposition $w = \sum_{i=1}^N w_i$, then

$$
\begin{aligned}
\langle E'(u^{k+1}), w\rangle &= \sum_{i=1}^N \langle E'(u^{k+1}), w_i\rangle = \sum_{i=1}^N \langle E'(u^{k+1}) - E'(v_i), w_i\rangle \quad \text{(FO)} \\
&= \sum_{i=1}^N \sum_{j=i+1}^N \langle E'(v_j) - E'(v_{j-1}), w_i\rangle \quad \text{(Telescope)} \\
&\le C_S \left(\sum_{i=1}^N \|e_i\|_{\mathcal{V}}^2\right)^{1/2} \left(\sum_{i=1}^N \|w_j\|_{\mathcal{V}}^2\right)^{1/2} \quad \text{(Strengthened CS, SS2)} \\
&\le C_S C_A \left(\sum_{i=1}^N \|e_i\|_{\mathcal{V}}^2\right)^{1/2} \|w\|_{\mathcal{V}}. \quad \text{(Stability, SS1)}
\end{aligned}
$$

Proof (3 of 3)

Then

$$d_{k+1} = E(u^{k+1}) - E(u) \le \frac{1}{2\mu}\|E'(u^{k+1})\|^2_{\mathcal{V}'} \quad \text{(Reciprocal UB)}$$

$$= \frac{1}{2\mu}\left(\sup_{0\neq w\in\mathcal{V}} \frac{\langle E'(u^{k+1}), w\rangle}{\|w\|_{\mathcal{V}}}\right)^2 \quad \text{(Operator Norm)}$$

$$\le \frac{1}{2\mu} C_S^2 C_A^2 \sum_{i=1}^{N} \|e_i\|^2_{\mathcal{V}}. \quad \text{(Step 2 Bound)}$$

□

Corollary (SSO Convergence)

Let u^k be the k-th iteration and $u^{k+1} = \mathrm{SSO}(u^k)$. Suppose that the space decomposition satisfies Assumptions (SS1) and (SS2) and the energy E satisfies Assumption (E1), then we have

$$E(u^{k+1}) - E(u) \le \rho(E(u^k) - E(u)), \quad \textit{with} \quad \rho = \frac{C_S^2 C_A^2}{C_S^2 C_A^2 + \mu^2}.$$

Proof.

Turn the Golden Key. □

Outline

Suppose we wish to solve the linear equation

$$\mathcal{L}_h(u) = f \quad \text{on the fine-grid space} \quad S_h$$

using corrections from a coarse-grid space $S_H \subset S_h$.

Two-Level Multigrid

Result: $u^{k+1} = \mathrm{MG}(u^k)$
$v_0 = u^k$;
Smooth (Linear Gauss-Seidel) on the fine grid: $v_1 = S(v_0)$;
Compute the coarse-grid defect:

$$d_H := R_h(f - \mathcal{L}_h(v_1)) \in S_H; \quad (R_h : S_h \searrow S_H) \tag{12}$$

Solve the coarse-grid correction problem: Find $s_H \in S_H$, such that

$$\mathcal{L}_H(s_H) = d_H \longrightarrow s_H \in S_H; \tag{13}$$

Apply the coarse-grid correction:

$$v_2 := v_1 + P_H s_H; \quad (P_H : S_H \nearrow S_h) \tag{14}$$

Smooth (Linear Gauss-Seidel) on the fine grid: $v_3 = S(v_2)$;
$u^{k+1} = v_3$;

Now, suppose we wish to solve the nonlinear equation

$$\mathcal{N}_h(u) = f \quad \text{on the fine-grid space} \quad S_h$$

using corrections from a coarse-grid space $S_H \subset S_h$.

Two-Level (Classical) Full Approximation Storage (FAS) Scheme

Result: $u^{k+1} = \mathrm{FAS}(u^k)$
$v_0 = u^k$;
Smooth (Nonlinear Gauss-Seidel) on the fine grid: $v_1 = S(v_0)$;
Compute the coarse-grid τ-*perturbation*:

$$\tau_H := \mathcal{N}_H(Q_h v_1) + R_h(f - \mathcal{N}_h(v_1)) \in S_H; \quad (Q_h, R_h : S_h \searrow S_H) \tag{15}$$

Solve the coarse-grid correction problem: Find $\eta_H \in S_H$, such that

$$\mathcal{N}_H(\eta_H) = \tau_H \longrightarrow s_H := \eta_H - Q_h v_1 \in S_H; \tag{16}$$

Apply the coarse-grid correction:

$$v_2 := v_1 + P_H s_H; \quad (P_H : S_H \nearrow S_h) \tag{17}$$

Smooth (Nonlinear Gauss-Seidel) on the fine grid: $v_3 = S(v_2)$;
$u^{k+1} = v_3$;

FASD and Subspace Energies

- Fast Subspace Descent (FASD) generalizes SSO and FAS.
- In FASD, we must create energies/operators for the subspace (coarse grid) problems. The subspace energy is denoted $E_i : \mathcal{V}_i \to \mathbb{R}$.
- E_i' plays the role $\mathcal{N}_H$.
- $E_i : \mathcal{V}_i \to \mathbb{R}$ could be quadratic, e.g., Jacobian-type, as in Newton's Method.
- E_i could be the natural restriction of E (the Galerkin condition) as in SSO.
- We will need a "nice" projection operator, for example the L^2 projection operator. We label this
$$Q_i : \mathcal{V} \to \mathcal{V}_i.$$
- We also need a canonical restriction operator:
$$R_i : \mathcal{V} \to \mathcal{V}_i,$$
the transpose of the natural embedding
$$I_i : \mathcal{V}_i \to \mathcal{V}.$$

Fast Subspace Descent (FASD) Algorithm

Result: $u^{k+1} = \mathrm{FASD}(u^k)$

$v_0 = u^k$;

for $i = 1 : N$ **do**

Compute the so-called subspace τ-*perturbation*:

$$\tau_i := E_i'(\xi_i) - R_i E'(v_{i-1}) \in \mathcal{V}_i', \quad \xi_i := Q_i v_{i-1}; \tag{18}$$

Solve the subspace correction problem: Find $\eta_i \in \mathcal{V}_i$, such that

$$\langle E_i'(\eta_i), w\rangle = \langle \tau_i, w\rangle, \quad \forall\, w \in \mathcal{V}_i \;\rightsquigarrow\; s_i := \eta_i - \xi_i \in \mathcal{V}_i; \tag{19}$$

Orthogonalize the subspace correction via line search:

$$\varepsilon_i := \alpha_i^* s_i, \quad \text{where} \quad \alpha_i^* = \operatorname*{argmin}_{\alpha\in\mathbb{R}} E(v_{i-1} + \alpha s_i); \tag{20}$$

Apply the subspace correction:

$$v_i := v_{i-1} + \varepsilon_i; \tag{21}$$

end

$u^{k+1} = v_N$;

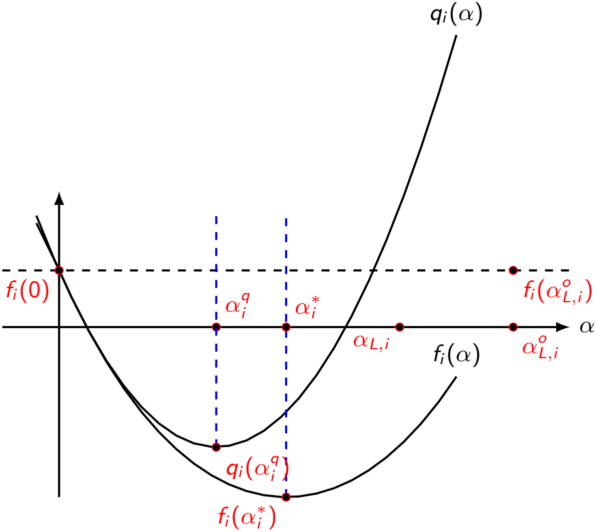

Figure: The energy section $f_i(\alpha) := E(v_{i-1} + \alpha s_i)$ and a quadratic approximation, q_i.

Generalizing SSO and FAS

Remark

We note that the FASD Algorithm generalizes the SSO Algorithm. They yield the same approximations in the case that

$$E_i(\eta) := E(v_{i-1} - Q_i v_{i-1} + \eta), \quad \forall\, \eta \in \mathcal{V}_i.$$

As a consequence of this choice, $\tau_i \equiv 0$ *and, for all* $w \in \mathcal{V}_i$*,*

$$\langle E'(v_{i-1} + s_i), w\rangle = \langle E'(v_{i-1} - Q_i v_{i-1} + \eta_i), w\rangle = \langle E_i'(\eta_i), w\rangle = 0.$$

With these choices in FASD, the line search (orthogonalization) is redundant.

Remark

The classical FAS algorithm of Achi Brandt is obtained by dropping the last (orthogonalization) step.

Theorem (FASD Lower Bound: Similar to SSO Lower Bound)

Suppose that E satisfies (E1), and let u^k be the k^{th} iteration in the FASD Algorithm. Then

$$E(u^k) - E(u^{k+1}) \geq \frac{\mu}{2} \sum_{i=1}^{N} \|\varepsilon_i\|_{\mathcal{V}}^2.$$

Proof.

$$\langle E'(v_i), w \rangle = 0, \quad w \in \mathrm{span}\{s_i\} =: \mathcal{W}. \quad \text{(orthogonality)}$$

Since $v_i - v_{i-1} = \varepsilon_i = \alpha_i^* s_i \in \mathrm{span}\{s_i\}$,

$$E(v_{i-1}) - E(v_i) \geq \frac{\mu}{2} \|v_{i-1} - v_i\|_{\mathcal{V}}^2 = \frac{\mu}{2} \|\varepsilon_i\|_{\mathcal{V}}^2. \quad \text{(Energy Trap for } f_i\text{)}$$

$$E(u^k) - E(u^{k+1}) = \sum_{i=1}^{N} \left(E(v_{i-1}) - E(v_i)\right) \geq \frac{\mu}{2} \sum_{i=1}^{N} \|\varepsilon_i\|_{\mathcal{V}}^2. \quad \text{(Telescope)}$$

□

Assumptions on Subspace Energies

(E3) (Strong convexity/Ellipticity:) There exists a constant μ_i such that for all $v, w \in \mathcal{V}_i$

$$\langle E_i'(w) - E_i'(v), w - v \rangle \geq \mu_i \| w - v \|_{\mathcal{V}}^2 .$$

(E4) (Lipschitz continuity of the first order derivative:) There exists a constant $L_i > 0$, such that

$$\| E_i'(w) - E_i'(v) \|_{\mathcal{V}'} \leq L_i \| w - v \|_{\mathcal{V}}$$

for all $w, v \in \mathcal{B}_i := Q_i \mathcal{B}^+$, where

$$\mathcal{B}^+ := \left\{ v \in \mathcal{V} \;\middle|\; \operatorname{dist}^2(v, \mathcal{B}) \leq \chi \right\}, \quad \text{(inflated ball)} \tag{22}$$

and χ is given by

$$\chi := \frac{2L^2}{\mu \min_i \mu_i^2} (E(u_0) - E(u)).$$

Quadratic Trap for E_i

Proposition

The sets $\mathcal{B}^+ \subseteq \mathcal{V}$ and $\mathcal{B}_i \subseteq \mathcal{V}_i$ are convex.

Lemma

Assume E_i satisfies assumptions (E3) and (E4). For any $v, w \in \mathcal{B}_i$,

$$\mu_i \left\| w - v \right\|_{\mathcal{V}}^2 \leq \langle E_i'(w) - E_i'(v), w - v \rangle \leq L_i \left\| w - v \right\|_{\mathcal{V}}^2 ,$$

and

$$\frac{\mu_i}{2} \left\| w - v \right\|_{\mathcal{V}}^2 + \langle E_i'(v), w - v \rangle \leq E_i(w) - E_i(v) \leq \langle E_i'(v), w - v \rangle + \frac{L_i}{2} \left\| w - v \right\|_{\mathcal{V}}^2 .$$

The lower bounds above hold for all $w \in \mathcal{V}_i$, without restriction.

Remark

If E_i is quadratic, then we can take $\mathcal{B}_i = \mathcal{V}_i$, and L_i, μ_i are simple.

Theorem (FASD Upper Bound)

Suppose the space decomposition satisfies (SS1) and (SS2), the energy E satisfies (E1) – (E2), and E_i satisfies (E3) – (E4). Then we have the upper bound

$$E(u^{k+1}) - E(u) \leq C_U \sum_{i=1}^{N} \|\varepsilon_i\|_{\mathcal{V}}^2,$$

where $C_U := C_A^2 \left[C_S + L\left(1 + \max_i\{L_i/\mu_i\}\right)\right]^2 / (2\mu)$.

Proof (1 of 3)

Note, for any $w \in \mathcal{V}$, we choose a stable decomposition $w = \sum_{i=1}^N w_i$, then

$$\langle E'(u^{k+1}), w\rangle = \sum_{i=1}^{N} \langle E'(u^{k+1}), w_i\rangle = \mathrm{I}_1 + \mathrm{I}_2,$$

where I_1 is similar to what we had in SSO and I_2 is new:

$$\mathrm{I}_1 := \sum_{i=1}^{N} \langle E'(u^{k+1}) - E'(v_i), w_i\rangle \quad \text{and} \quad \mathrm{I}_2 := \sum_{i=1}^{N} \langle E'(v_i), w_i\rangle.$$

Proof (2 of 3)

Using the stability of the decomposition (SS1) and the strengthened Cauchy-Schwartz inequality (SS2), I_1 can be estimated in exactly the same way as in the convergence proof for SSO. Therefore,

$$\mathrm{I}_1 \leq C_S C_A \left(\sum_{i=1}^{N} \|\varepsilon_i\|_{\mathcal{V}}^2 \right)^{1/2} \|w\|_{\mathcal{V}}. \quad \text{(SSO, Done)}$$

For I_2, we insert $\tau_i - E_i'(\xi_i + s_i)$, which is zero in $\mathcal{V}_i'$, use the Lipschitz continuities to get

$$\begin{aligned} \mathrm{I}_2 &= \sum_{i=1}^{N} \langle E'(v_i) - E'(v_{i-1}) - E_i'(\xi_i + s_i) + E_i'(\xi_i), w_i \rangle \\ &\leq \sum_{i=1}^{N} \left(L\|\varepsilon_i\|_{\mathcal{V}} + L_i\|s_i\|_{\mathcal{V}} \right) \|w_i\|_{\mathcal{V}}, \end{aligned}$$

provided $\xi_i + s_i$ and ξ_i stay in $\mathcal{B}_i$, which can be shown.

Proof (3 of 3)

Recall $\varepsilon_i = \alpha_i^* s_i$. In a key technical lemma, we can show that $\frac{\mu_i}{L} \le \alpha_i^*$. Thus,

$$\mathrm{I}_2 \le \sum_{i=1}^{N} \left(L\|\varepsilon_i\|_{\mathcal{V}} + \frac{L_i}{\alpha_i^*}\|\varepsilon_i\|_{\mathcal{V}} \right) \|w_i\|_{\mathcal{V}} \le L \sum_{i=1}^{N} \left(1 + \frac{L_i}{\mu_i}\right) \|\varepsilon_i\|_{\mathcal{V}} \|w_i\|_{\mathcal{V}}$$

$$\le L \left(1 + \max_{i=1}^{N} \frac{L_i}{\mu_i}\right) \left(\sum_{i=1}^{N} \|\varepsilon_i\|_{\mathcal{V}}^2 \right)^{1/2} \left(\sum_{i=1}^{N} \|w_i\|_{\mathcal{V}}^2 \right)^{1/2} \qquad \text{(Discrete CS)}$$

$$\le L C_A \left(1 + \max_{i=1}^{N} \{L_i/\mu_i\}\right) \left(\sum_{i=1}^{N} \|\varepsilon_i\|_{\mathcal{V}}^2 \right)^{1/2} \|w\|_{\mathcal{V}}. \quad \text{(Stability, SS1)}$$

Putting the I_1 and I_2 estimates together, we have, for any $w \in \mathcal{V}$,

$$\langle E'(u^{k+1}), w \rangle \le C_A \left[C_S + L \left(1 + \max_{i=1}^{N} \frac{L_i}{\mu_i}\right) \right] \left(\sum_{i=1}^{N} \|\varepsilon_i\|_{\mathcal{V}}^2 \right)^{1/2} \|w\|_{\mathcal{V}},$$

From here the result follows as in SSO.

Corollary (Convergence of FASD)

Let u^k be the k-th iteration and $u^{k+1} = \mathrm{FASD}(u^k)$. Suppose that the space decomposition satisfies Assumptions (SS1) and (SS2), the energy E satisfies Assumption (E1) – (E2), and the energy E_i satisfies Assumption (E3) – (E4), then we have

$$E(u^{k+1}) - E(u) \leq \rho(E(u^k) - E(u)),$$

with

$$\rho = \frac{C_A^2 \left[C_S + L\left(1 + \max_i\{L_i/\mu_i\}\right)\right]^2}{C_A^2 \left[C_S + L\left(1 + \max_i\{L_i/\mu_i\}\right)\right]^2 + \mu^2}.$$

Furthermore if E_i is the quadratic energy ($L_i = \mu_i = 1$)

$$E_i(w) = \frac{1}{2}\|w - \xi_i\|_{\mathcal{V}}^2 = \frac{1}{2}\|w - Q_i v_{i-1}\|_{\mathcal{V}}^2, \quad \forall\, w \in \mathcal{V}_i, \tag{23}$$

then

$$\rho = \frac{C_A^2 \left(C_S + 2L\right)^2}{C_A^2 \left(C_S + 2L\right)^2 + \mu^2}.$$

Proof.

Turn the Golden Key. □

A Brief Summary

- This seems great.
- FASD can be a lot cheaper that SSO.
- It looks more like FAS and allows for a lot of flexibility.
- But there is a problem.
- FASD isn't FAS; there is an extra, potentially expensive line search step at the end.
- Can this be eliminated?

Outline

FASD With Approximate Line Search (FASD-ALS)

Result: $u^{k+1} = \mathrm{FASD} - \mathrm{ALS}(u^k)$

$v_0 = u^k$;

for $i = 1 : N$ **do**

Compute the subspace τ-*perturbation*: let $\xi_i = Q_i v_{i-1}$ and

$$\tau_i := E_i'(\xi_i) - R_i E'(v_{i-1}) \in \mathcal{V}_i'; \tag{24}$$

Solve the subspace correction problem: Find $\eta_i \in \mathcal{V}_i$, such that

$$\langle E_i'(\eta_i), w\rangle = \langle \tau_i, w\rangle, \ \forall w \in \mathcal{V}_i, \ \rightsquigarrow \ s_i := \eta_i - \xi_i \in \mathcal{V}_i, \tag{25}$$

Apply the subspace correction using quadratic (approximate) step size:

$$v_i := v_{i-1} + \alpha_i^q s_i, \quad \alpha_i^q := -\frac{\langle R_i E'(v_{i-1}), s_i\rangle}{L\|s_i\|_{\mathcal{V}}^2}. \tag{26}$$

end

$u^{k+1} := v_N$;

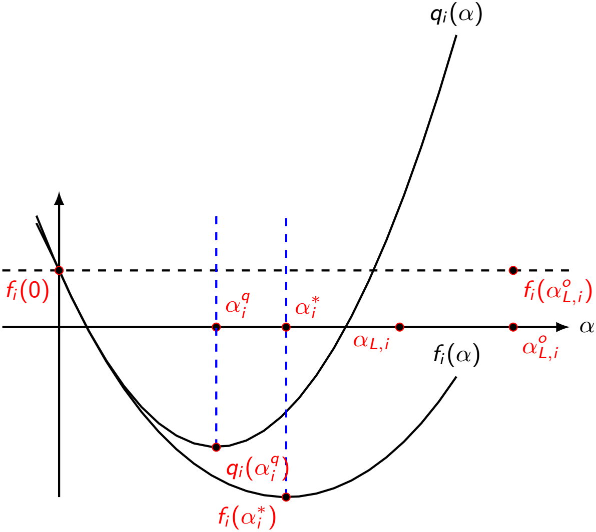

Figure: The function $f_i(\alpha) := E(v_{i-1} + \alpha s_i)$ and a quadratic approximation, q_i.

Convergence Analysis Details

- Now, the last orthogonalization condition is broken. So the lower bound is more challenging.
- We have opted instead to use a quadratic line search approximation.
- The quadratic line search approximation can be much more efficient.
- For the upper bound, the analysis is almost identical with that of the FASD method. We only need a lower bound for α_i^q, which is easily obtained.

Corollary (Convergence of FASD-ALS)

Let u^k be the k-th iteration and $u^{k+1} = \mathrm{FASD} - \mathrm{ALS}(u^k)$. Suppose that the space decomposition satisfies Assumptions (SS1) and (SS2), the energy E satisfies Assumptions (E1) – (E2), and the energy E_i satisfies Assumptions (E3) – (E4), then we have

$$E(u^{k+1}) - E(u) \le \rho(E(u^k) - E(u)),$$

with

$$\rho = \frac{C_A^2\left[C_S + L\left(1 + \max_i\{L_i/\mu_i\}\right)\right]^2}{C_A^2\left[C_S + L\left(1 + \max_i\{L_i/\mu_i\}\right)\right]^2 + L\mu},$$

where

$$C_U = C_A^2\left[C_S + L\left(1 + \max_i\{L_i/\mu_i\}\right)\right]^2/(2\mu) \quad \text{and} \quad C_L = \frac{L}{2}$$

Remark

As before, we get some simplification, $L_i = \mu_i = 1$, using

$$E_i(w) = \frac{1}{2}\|w - \xi_i\|_{\mathcal{V}}^2.$$

Full Approximation Storage (FAS) = FASD With No Line Search

Result: $u^{k+1} = \mathrm{FAS}(u^k)$

$v_0 = u^k$;

for $i = 1 : N$ **do**

Compute the subspace τ-*perturbation*: let $\xi_i = Q_i v_{i-1}$ and

$$\tau_i := E_i'(\xi_i) - R_i E'(v_{i-1}) \in \mathcal{V}_i'; \tag{27}$$

Solve the subspace correction problem: Find $\eta_i \in \mathcal{V}_i$, such that

$$\langle E_i'(\eta_i), w\rangle = \langle \tau_i, w\rangle, \ \forall w \in \mathcal{V}_i, \ \rightsquigarrow \ s_i := \eta_i - \xi_i \in \mathcal{V}_i. \tag{28}$$

Apply the subspace correction using step size of 1:

$$v_i := v_{i-1} + s_i. \tag{29}$$

end

$u^{k+1} := v_N$;

Convergence Analysis Details

- Again, the orthogonalization is broken. The lower bound is harder to prove, requiring a few more technical lemmas.
- For the lower bound, we must require more from the subspace energy E_i, namely, some approximation property:

Approximation Property

(AP) Both E and E_i are twice Fréchet differentiable. Furthermore, there exists a constant $\epsilon < \mu/2$ so that for all $w \in \mathcal{B}^+$ and all $u_i, v_i \in \mathcal{V}_i$

$$|\langle E''(w)u_i, v_i\rangle - \langle E_i''(Q_i w)u_i, v_i\rangle| \le \epsilon \|u_i\|_{\mathcal{V}} \|v_i\|_{\mathcal{V}}.$$

- The upper bound proof is similar to FASD.

Corollary (Convergence of FAS)

Let u^k be the k-th iteration and $u^{k+1} = \mathrm{FAS}(u^k)$. Suppose that the space decomposition satisfies Assumptions (SS1) and (SS2), the energy E satisfies Assumption (E1) – (E2), and the energy E_i satisfies Assumption (AP) with $\epsilon < \mu/2$, then we have

$$E(u^{k+1}) - E(u) \le \rho(E(u^k) - E(u)),$$

with

$$\rho = \frac{(C_S+\epsilon)^2 C_A^2}{(C_S+\epsilon)^2 C_A^2 + \mu(\mu - 2\epsilon)},$$

where

$$C_U = C_A^2 (C_S+\epsilon)^2/(2\mu) \quad \text{and} \quad C_L = \frac{\mu}{2} - \varepsilon.$$

Outline

A Nonlinear Problem

Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded open set, with a sufficiently regular boundary. We consider the following problem: given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$\left(|u|^{p-2}u, \xi\right) + \varepsilon^2 \left(\nabla u, \nabla \xi\right) = \left(f, \xi\right), \quad \forall\, \xi \in H_0^1(\Omega), \tag{30}$$

where $2 \leq p < \infty$, when $d = 2$ and $2 \leq p \leq 6$, when $d = 3$, and $\varepsilon > 0$ is a parameter.

Theorem

Suppose p is restricted as above. For any $\nu \in H_0^1(\Omega)$, define the energy

$$E(\nu) := \frac{1}{p}\left\|\nu\right\|_{L^p}^p + \frac{\varepsilon^2}{2}\left\|\nabla \nu\right\|^2 - \left(f, \nu\right), \quad p \geq 2. \tag{31}$$

This functional is twice Fréchet differentiable; satisfies (E1) and (E2); and is strictly convex, and coercive. Therefore, it has a unique global minimizer. Furthermore, $u \in H_0^1(\Omega)$ is the unique minimizer of (31) iff it is the solution of (30).

Finite Element Approximation in 2D

- Now, suppose that $\Omega \subset \mathbb{R}^2$ is a polygonal domain and $\mathcal{T}_H$ is a conforming triangulation of Ω.
- Let $\mathcal{T}_h$ be the triangulation obtained by quadri-secting $\mathcal{T}_h$.
- Define

$$S_h := \left\{ v \in C(\Omega) \cap H_0^1(\Omega) \middle| v|_K \in \mathcal{P}_1(K), \ \forall K \in \mathcal{T}_h \right\}.$$

  With a similar definition for S_H. Then, $S_H \subset S_h$, and the containment is proper.
- We shall consider the minimization of energy E restricted to S_h , which is a subspace of $H_0^1(\Omega)$ (the Ritz Approximation),

$$\min_{v \in S_h} E(v),$$

  and thus now $\mathcal{V} = S_h$ (finite dim.) with norm $|v|_1 = \|\nabla v\|$. Notice that (E1) and (E2) still hold, as $S_h \subset H_0^1(\Omega)$.

A Two-Level Subspace Decomposition

- Let $\mathcal{N} = \{\mathbf{x}_i\}_{i=1}^{N} \subset \mathbb{R}^2$ be the set of *interior* nodes of $\mathcal{T}_h$ and define the Lagrange nodal basis

$$B_h = \{\psi_i \in S_h \mid \psi_i(\mathbf{x}_j) = \delta_{i,j},\ 1 \leq i, j \leq N\}.$$

B_h is a *bona fide* basis for S_h, and we may use the following decomposition

$$\mathcal{V} = \sum_{i=0}^{N} \mathcal{V}_i = S_h, \tag{32}$$

where $\mathcal{V}_0 = S_H$, $\mathcal{V}_i = \mathrm{span}(\{\psi_i\})$, $1 \leq i \leq N$.

Theorem (Subspace Decomposition Satisfies SS1 and SS2)

The decomposition of the finite element space S_h described in (32) satisfies Assumption (SS1), and $C_A > 0$ is independent of N and h. Furthermore, let E be defined as in (31). Then Assumption (SS2) holds, and C_S is independent of N and h.

Contraction Factors: FAS

Table: Numerical results of FAS (varying p and ε, fixed $h = 1/64$)

FAS	$\varepsilon^2 = 1$	$\varepsilon^2 = 1/2$	$\varepsilon^2 = 1/4$	$\varepsilon^2 = 1/8$	$\varepsilon^2 = 10^{-1}$	$\varepsilon^2 = 10^{-2}$	$\varepsilon^2 = 10^{-3}$
$p = 4$	15 (0.195)	15 (0.193)	14 (0.189)	14 (0.186)	14 (0.186)	12 (0.164)	10 (0.133)
$p = 5.5$	14 (0.195)	14 (0.192)	14 (0.189)	14 (0.189)	14 (0.189)	12 (0.166)	11 (0.162)
$p = 6$	15 (0.195)	15 (0.192)	14 (0.190)	14 (0.190)	14 (0.189)	13 (0.167)	11 (0.167)
$p = 8$	15 (0.196)	15 (0.193)	15 (0.192)	14 (0.191)	14 (0.190)	13 (0.176)	12 (0.173)
$p = 10$	15 (0.198)	15 (0.196)	15 (0.194)	15 (0.192)	14 (0.191)	13 (0.178)	12 (0.170)
$p = 20$	16 (0.216)	16 (0.221)	16 (0.210)	15 (0.197)	15 (0.194)	14 (0.182)	13 (0.178)
$p = 40$	18 (0.267)	18 (0.273)	17 (0.248)	16 (0.209)	16 (0.204)	14 (0.188)	13 (0.180)
$p = 80$	21 (0.333)	21 (0.338)	20 (0.304)	18 (0.243)	17 (0.226)	15 (0.192)	14 (0.200)

- Original FAS.
- We consider standard multilevel nodal based space decomposition $\mathcal{V} = \sum_{\ell=1}^{J} \sum_{i=1}^{N_\ell} \text{span}\{\phi_i^\ell\}$.
- E_i is defined as the restriction of E on the subspace $\mathcal{V}_{\ell,i} := \text{span}\{\phi_i^\ell\}$.
- Newton's method is used to solve the local nonlinear problem. Typically, less than 5 iterations are needed for solving the local problems in all of our numerical tests.

Contraction Factors: FASq1

Table: Numerical results of FASq1 (varying p and ε, fix $h = 1/64$)

FASq1	$\varepsilon^2 = 1$	$\varepsilon^2 = 1/2$	$\varepsilon^2 = 1/4$	$\varepsilon^2 = 1/8$	$\varepsilon^2 = 10^{-1}$	$\varepsilon^2 = 10^{-2}$	$\varepsilon^2 = 10^{-3}$
$p = 4$	15 (0.193)	15 (0.189)	14 (0.185)	14 (0.180)	13 (0.179)	23 (0.331)	-
$p = 5.5$	15 (0.192)	15 (0.189)	14 (0.186)	14 (0.184)	14 (0.183)	-	-
$p = 6$	15 (0.192)	15 (0.189)	14 (0.187)	14 (0.185)	14 (0.183)	-	-
$p = 8$	15 (0.193)	15 (0.190)	14 (0.190)	14 (0.191)	14 (0.186)	-	-
$p = 10$	15 (0.195)	15 (0.193)	14 (0.191)	14 (0.192)	14 (0.187)	-	-
$p = 20$	16 (0.211)	16 (0.215)	16 (0.215)	16 (0.216)	16 (0.220)	-	-
$p = 40$	18 (0.260)	18 (0.281)	19 (0.298)	21 (0.334)	23 (0.367)	-	-
$p = 80$	21 (0.342)	23 (0.383)	25 (0.407)	109 (0.844)	-	-	-

- FASq1 details.
- We use the quadratic subspace energy

$$E_{\ell,i}(w) = \frac{1}{2}\|w - \xi_{\ell,i}\|_{\mathcal{V}}^2 = \frac{1}{2}\|w - Q_{\ell,i}v_{\ell,i-1}\|_{\mathcal{V}}^2, \quad \forall\, w \in \mathcal{V}_{\ell,i};$$

- Use the multilevel nodal based space decomposition $\mathcal{V} = \sum_{\ell=1}^{J}\sum_{i=1}^{N_\ell}\mathcal{V}_{\ell,i}$.
- We skip the line-search/orthogonalization step.

Contraction: FASq2

Table: Numerical results of FASq2 (varying p and ε, fix $h = 1/64$)

FASq2	$\varepsilon^2 = 1$	$\varepsilon^2 = 1/2$	$\varepsilon^2 = 1/4$	$\varepsilon^2 = 1/8$	$\varepsilon^2 = 10^{-1}$	$\varepsilon^2 = 10^{-2}$	$\varepsilon^2 = 10^{-3}$
$p = 4$	14 (0.190)	14 (0.187)	14 (0.183)	14 (0.181)	14 (0.181)	-	-
$p = 5.5$	14 (0.189)	14 (0.189)	14 (0.183)	14 (0.185)	14 (0.187)	-	-
$p = 6$	14 (0.188)	14 (0.186)	14 (0.185)	14 (0.188)	14 (0.190)	-	-
$p = 8$	14 (0.190)	14 (0.190)	14 (0.188)	14 (0.193)	15 (0.196)	-	-
$p = 10$	15 (0.191)	15 (0.191)	15 (0.193)	15 (0.199)	15 (0.202)	-	-
$p = 20$	15 (0.211)	16 (0.223)	17 (0.239)	18 (0.265)	20 (0.290)	-	-
$p = 40$	18 (0.264)	19 (0.300)	21 (0.334)	29 (0.452)	49 (0.643)	-	-
$p = 80$	21 (0.350)	24 (0.393)	32 (0.504)	-	-	-	-

- FASq2 details.
- We use space decomposition $\mathcal{V} = \sum_{\ell=1}^{J} \mathcal{V}^\ell$.
- We use the simple quadratic E_i.
- Corrections are computed by inverting an SPD matrix defined on $\mathcal{V}^\ell$. For our example, this is equivalent to solving a discrete Laplacian matrix on each level, which is still expensive.
- Therefore, we solve the discrete Laplacian matrix approximately by just applying one step of symmetric Gauss-Seidel method.

Complexity

Table: Computational complexity comparison with $\varepsilon = 1$ and $p = 6$

	FAS		FASq2	
h	#iter	CPU time	#iter	CPU time
1/32	15	1.65	14	0.03
1/64	15	7.86	14	0.05
1/128	16	45.60	14	0.16
1/256	16	391.08	15	0.49
1/512	16	>1,000	15	1.67
1/1024	16	>1,000	15	7.12

Outline

Concluding Remarks

1. We have proven that a generalization of FAS, FASD, converges globally and geometrically.
2. Proofs are based on viewing FASD as an inexact SSO method.
3. In the finite dimensional case, this convergence does not deteriorate as $h \to 0$ (i.e., as the degrees of freedom, N, increase).
4. The complexity of FASD/FAS can be significantly less than SSO.
5. Convergence of the classical FAS method requires an extra approximation assumption.
6. Everything works well for the second-order nonlinear problem. The Stationary Cahn-Hilliard Equation is a bit more delicate.
7. The difficulty in the CH setting is dealing with the negative norms. This FASD theory will need to be extended to the mixed (saddle point) setting to achieve a truly efficient method.

Thanks. Questions?

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