The Cahn-Hilliard equation in image inpainting

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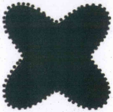
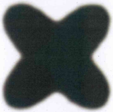
NSF-CBMS Conference
The Cahn–Hilliard equation : recent advances and applications

Applications of the Cahn-Hilliard equation in image processing :

- Image denoising
- Image inpainting



(a) $t = 0$



(b) $t = 5 \cdot 10^{-8}$

(c) $t = 5 \cdot 10^{-7}$

(d) $t = 5 \cdot 10^{-6}$

(e) $t = 2.5 \cdot 10^{-5}$

(f) $t = 5 \cdot 10^{-5}$

(a) $t = 0$

(b) $t = 2.5 \cdot 10^{-9}$

(c) $t = 5 \cdot 10^{-9}$

(d) $t = 10^{-8}$

(e) $t = 2.5 \cdot 10^{-8}$

(f) $t = 5 \cdot 10^{-8}$

A Cahn-Hilliard model for binary image inpainting (A. Bertozzi-S. Esedoglu-A. Gillette, 2007)

Image inpainting : consists in filling in parts of an image/video from the surrounding area (interpolation)

Applications : restoration of old paintings, removing scratches, altering scenes, restoration of motion pictures, ...

PDE's for image inpainting : M. Bertalmio et al. (Navier-Stokes like model)

Other models : second-order models (S. Esedoglu-J. Shen)


Since 1699, when French explorers
landed at the great bend of the
Mississippi River and celebrated
the first Mardi Gras in North America,
New Orleans has brewed a fascinating
melange of cultures. It was French,
then Spanish, then French again, then
sold to the United States. Through all
these years, and even into the 1900s,
others arrived from everywhere:
Acadians (Cajuns), Africans, indige-

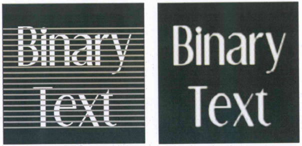
$$g(x,s) = \lambda_0(s - h(x))\chi_{\Omega\setminus D}(x),\ \lambda_0 > 0,\ D \subset\subset \Omega$$

Equation :

$$\frac{\partial u}{\partial t} + \epsilon\Delta^2 u - \frac{1}{\epsilon}\Delta f(u) + g(x,u) = 0,\ \epsilon > 0$$

$h(x)$: given image ($h \in L^2(\Omega)$)
D : inpainting domain (damaged region)
$f(s) = 4s^3 - 6s^2 + 2s$
$g(x,u)$: added to keep u close to the image $h(x)$ outside the inpainting region (fidelity term)

Advantages (over, e.g., $u = h$ outside D) : no regularity assumption on D, no perfect h outside D



(a) (b)

Algorithm : dynamic two-steps algorithm involving ϵ

First step : large value of ϵ to connect the edges

Second step : small value of ϵ (depending on the mesh size) ; solution obtained in the first step as initial datum

Idea : solve the equation to steady state to construct an inpainting version $u(x)$ of $h(x)$

Advantage : fast in numerical simulations

Remark : Limit problem when $\lambda_0 = +\infty$ (g of class $\mathcal{C}^2$) :

$$\Delta(\epsilon\Delta u - \frac{1}{\epsilon}f(u)) = 0 \text{ in } D$$

$$u = h \text{ on } \partial D$$

$$\nabla u = \nabla h \text{ on } \partial D$$

$\rightarrow$ Continuation of the image gradient into the missing domain

$\rightarrow$ λ_0 large in numerical simulations

Mathematical analysis (L. Cherfils-H. Fakih-A. Miranville, IPI, SIIMS) :

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + \chi_{\Omega \setminus D}(x) u = 0$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma$$
$$u\big|_{t=0} = u_0$$

We take $f(s) = s^3 - s$ (more generally : $f(s) = \sum_{i=1}^{2p+1} a_i s^i$, $a_{2p+1} > 0$), $h \equiv 0$

First well-posedness result : A. Bertozzi et al.

To go further : global in time/dissipative estimate

First step : obtain an estimate in $H^{-1}(\Omega)$

$\rightarrow$ We need to estimate $\langle u \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega u\, dx$

Classical Cahn-Hilliard equation : conservation of mass

If $|\langle u_0 \rangle| \leq M$, then $|\langle u(t) \rangle| \leq M$, $t \geq 0$

Equation for $\langle u \rangle$:

$$\frac{d\langle u \rangle}{dt} + \frac{1}{\mathrm{Vol}(\Omega)} \int_{\Omega \backslash D} u\, dx = 0$$

We set

$$u = \langle u \rangle + v$$

We find

$$\frac{d\langle u \rangle}{dt} + c_0 \langle u \rangle = -\frac{1}{\mathrm{Vol}(\Omega)} \int_{\Omega \backslash D} v \, dx, \; c_0 = \frac{\mathrm{Vol}(\Omega \backslash D)}{\mathrm{Vol}(\Omega)}$$

v is solution to

$$\begin{array}{l} \frac{\partial}{\partial t}(-\Delta)^{-1} v - \Delta v + f(\langle u \rangle + v) - \langle f(\langle u \rangle + v) \rangle \\ +(-\Delta)^{-1}(\chi_{\Omega \backslash D}(x) u - \langle \chi_{\Omega \backslash D}(x) u \rangle) = 0 \end{array}$$

$(-\Delta)^{-1}$: inverse minus Laplacian acting on functions with null average $(\langle v \rangle = 0)$

Multiply the equation by v

Use the inequality

$$
\begin{aligned}
&((f(\langle u\rangle + v) - \langle f(\langle u\rangle + v)\rangle, v))_{L^2} \\
&= ((f(\langle u\rangle + v) - f(\langle u\rangle), v))_{L^2} \\
&\geq \tfrac{c_0}{2} \int_\Omega (v^4 + v^2\langle u\rangle^2)\, dx - \|v\|_{L^2}^2
\end{aligned}
$$

We obtain

$$
\frac{d}{dt}\|v\|_{H^{-1}}^2 + \|\nabla v\|_{L^2}^2 + c_0 \int_\Omega (v^4 + v^2\langle u\rangle^2)\, dx \leq c
$$

Consequence : $\|v\|_{H^{-1}}^2 \leq e^{-ct}\|v_0\|_{H^{-1}}^2 + c',\ c > 0,\ t \geq 0$

Multiply the equation by $-\Delta v$:

$$\|v(t)\|_{L^2} \leq Q(\|u_0\|_{L^2}),\ t \geq 0$$
$$\|v(t)\|_{L^2} \leq c,\ t \geq t_0,\ t_0 > 0$$

Q : monotone increasing function
c : independent of u_0 and t, $\|u_0\|_{L^2} \leq R$, $t_0 = t_0(R)$

Equation for $\langle u \rangle$:

$$\frac{d\langle u \rangle}{dt} + c_0 \langle u \rangle = -\frac{1}{\mathrm{Vol}(\Omega)} \int_{\Omega \backslash D} v \, dx$$

$\rightarrow |\langle u(t) \rangle| \leq Q(\|u_0\|_{L^2}) e^{-ct} + c',\ c > 0,\ t \geq 0$

Well-posedness, further regularity

Existence of finite-dimensional attractors

Open problem : convergence of solutions to steady states

Numerical simulations : one-step algorithm with threshold

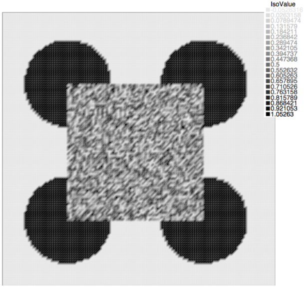
Two-step algorithm : $\epsilon = 0.1$ and then $\epsilon = 0.01$

Here : $\epsilon = 0.05$ and then threshold

If $u \geq \frac{1}{2}$, then we take $u = 1$

If $u < \frac{1}{2}$, then we take $u = 0$

When D is not "too large" : results comparable with the two-steps algorithm, computation time divided by two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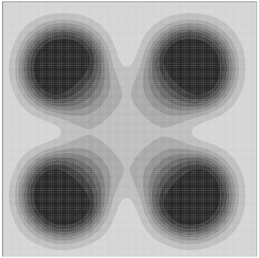

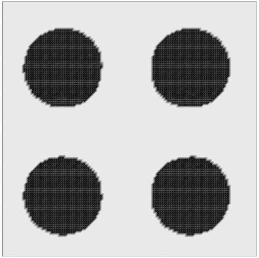

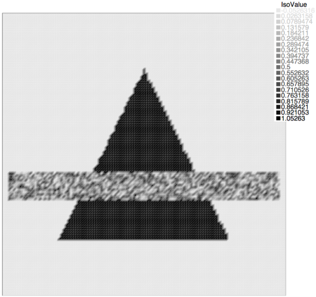
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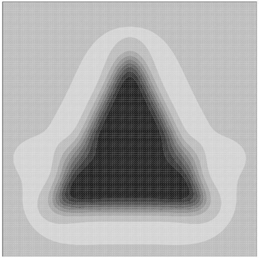
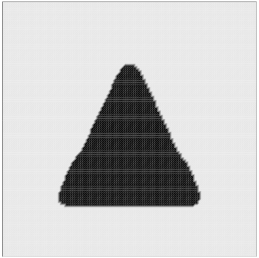




Logarithmic nonlinear terms :

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + \chi_{\Omega\backslash D}(x)u = 0\\
&\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma\\
&u\big|_{t=0} = u_0
\end{aligned}$$

$f(s) = -2\theta_0 s + \theta_1 \ln \frac{1+s}{1-s},\ s \in (-1,1),\ 0 < \theta_1 < \theta_0$

For $h \neq 0$, we need $\int_{\Omega\backslash D} h\, dx = 0$

We have a local (in time) existence result

Theorem : We assume that $u_0 \in H^1(\Omega)$, $|\langle u_0 \rangle| < 1$ and $-1 < u_0(x) < 1$ a.e. $x \in \Omega$. Then, there exists $T_0 = T_0(u_0)$ and a solution u such that $u \in \mathcal{C}([0,T_0]; H^{-1}(\Omega)) \cap L^\infty(0,T_0; H^1(\Omega)) \cap L^2(0,T_0; H^2(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2(0,T_0; H^{-1}(\Omega))$. Furthermore,

$$-1 < u(t,x) < 1 \text{ a.e. } (t,x) \in (0,T_0) \times \Omega.$$

We approximate the singular nonlinear term by regular ones

The approximated functions need to satisfy a (uniform) coercivity relation

We set $F_N(s) = F_{1,N}(s) - \theta_0 s^2$

$$F_{1,N}(s) = \begin{cases} \sum_{k=0}^{4} \frac{1}{k!} F_1^{(k)}(1 - \frac{1}{N})(s - 1 + \frac{1}{N})^k, \ s \geq 1 - \frac{1}{N}, \\ F_1(s), \ |s| \leq 1 - \frac{1}{N}, \\ \sum_{k=0}^{4} \frac{1}{k!} F_1^{(k)}(-1 + \frac{1}{N})(s + 1 - \frac{1}{N})^k, \ s \leq -1 + \frac{1}{N}. \end{cases}$$

$F(s) = F_1(s) - \theta_0 s^2$

$$F_{1,N}(s) = \sum_{k=0}^{4} \frac{1}{k!} F_1^{(k)}(1 - \frac{1}{N})(s - 1 + \frac{1}{N})^k, \ s > 1 - \frac{1}{N}$$

$$F_{1,N}(s) = F_1(s), \ |s| \leq 1 - \frac{1}{N}$$

$$F_{1,N}(s) = \sum_{k=0}^{4} \frac{1}{k!} F_1^{(k)}(-1 + \frac{1}{N})(s + 1 - \frac{1}{N})^k, \ s < -1 + \frac{1}{N}$$

The approximated functions $f_N = F'_N$ satisfy

- $F_N \in \mathcal{C}^4(R)$
- $f_N(0) = 0$
- $f'_N \geq -\theta_0, F_N \geq -c_1, c_1 \geq 0$
- $f_N(s)s \geq c_2(F_N(s) + |f_N(s)|) - c_3,\ c_2 > 0,\ c_3 \geq 0,\ s \in R$
- $(f_N(s+a) - f_N(a))s \geq c_4(s^4 + a^2 s^2) - c_5,\ c_4 > 0,\ c_5 \geq 0,\ s,\ a \in R$

All the constants are independent of N

Approximated problems :

$$\begin{cases} \frac{\partial u_N}{\partial t} + \Delta^2 u_N - \Delta f_N(u_N) + \chi_{\Omega \setminus D}(x)(u_N - h) = 0 \\ \frac{\partial u_N}{\partial \nu} = \frac{\partial \Delta u_N}{\partial \nu} = 0 \text{ on } \Gamma \\ u_N\big|_{t=0} = u_0 \end{cases}$$

We have the well-posedness and the regularity of the solutions to the approximated problems

A priori estimates :

Equation for the spatial average :

$$\frac{d\langle u_N\rangle}{dt} + \frac{1}{\mathrm{Vol}(\Omega)}\int_{\Omega\setminus D} u_N\,dx = 0$$

Set $u_N = \langle u_N\rangle + \overline{u}_N$:

$$\frac{d\langle u_N\rangle}{dt} + c\langle u_N\rangle = -\frac{1}{\mathrm{Vol}(\Omega)}\int_{\Omega\setminus D} \overline{u}_N\,dx$$

$c = \frac{\mathrm{Vol}(\Omega\setminus D)}{\mathrm{Vol}(\Omega)}$

$\overline{u}_N$ is solution to

$$\frac{\partial \overline{u}_N}{\partial t} + \Delta^2 \overline{u}_N - \Delta(f_N(u_N) - \langle f_N(u_N) \rangle) + \chi_{\Omega\setminus D}(x) u_N - \langle \chi_{\Omega\setminus D}(x) u_N \rangle = 0$$

$$\frac{\partial \overline{u}_N}{\partial \nu} = \frac{\partial \Delta \overline{u}_N}{\partial \nu} = 0 \text{ on } \Gamma$$

$$\overline{u}_N|_{t=0} = v_0 = u_0 - \langle u_0 \rangle$$

Equivalent formulation :

$$(-\Delta)^{-1} \frac{\partial \overline{u}_N}{\partial t} - \Delta \overline{u}_N + f_N(u_N) - \langle f_N(u_N) \rangle$$

$$+ (-\Delta)^{-1} (\chi_{\Omega\setminus D}(x) u_N - \langle \chi_{\Omega\setminus D}(x) u_N \rangle) = 0$$

$$\frac{\partial \overline{u}_N}{\partial \nu} = 0 \text{ on } \Gamma$$

Multiply by $\overline{u}_N$:

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}_N\|_{-1}^2 + \|\overline{u}_N\|_V^2$$

$$+((f_N(u_N) - \langle f_N(u_N)\rangle, \overline{u}_N)) + ((\chi_{\Omega\setminus D}(x)u_N, (-\Delta)^{-1}\overline{u}_N)) = 0$$

Note that

$$((f_N(u_N) - \langle f_N(u_N)\rangle, \overline{u}_N)) = ((f_N(u_N) - f_N(\langle u_N\rangle), \overline{u}_N))$$

Thus :

$$((f_N(u_N) - \langle f_N(u_N)\rangle, \overline{u}_N)) \geq c_4(\|\overline{u}_N\|_{L^4(\Omega)}^4 + \langle u_N\rangle^2\|\overline{u}_N\|^2) - c$$

Furthermore :

$$|((\chi_{\Omega\setminus D}(x)u_N, (-\Delta)^{-1}\overline{u}_N))| \leq c(\|\overline{u}_N\|^2 + |\langle u_N\rangle|\|\overline{u}_N\|)$$

$$\leq \frac{c_4}{2}(\|\overline{u}_N\|_{L^4(\Omega)}^4 + \langle u_N\rangle^2\|\overline{u}_N\|^2) + c$$

Thus :

$$\frac{d}{dt}\|\overline{u}_N\|_{-1}^2 + \|\overline{u}_N\|_V^2 + c_4(\|\overline{u}_N\|_{L^4(\Omega)}^4 + \langle u_N\rangle^2\|\overline{u}_N\|^2) \leq c$$

Next :

$$\frac{d\langle u_N\rangle^2}{dt} + c_6\langle u_N\rangle^2 \leq c\|\overline{u}_N\|^2$$

Thus :

$$\frac{d\langle u_N\rangle^2}{dt} + c_6\langle u_N\rangle^2 \leq \frac{c_4}{2}(\|\overline{u}_N\|_{L^4(\Omega)}^4 + \langle u_N\rangle^2\|\overline{u}_N\|^2) + c$$

Sum :

$$\frac{dE_{1,N}}{dt} + c(\|u_N\|_{H^1(\Omega)}^2 + \|\overline{u}_N\|_{L^4(\Omega)}^4 + \langle u_N\rangle^2\|\overline{u}_N\|^2) \leq c',\ c > 0$$

$$E_{1,N} = \langle u_N\rangle^2 + \|\overline{u}_N\|_{-1}^2$$

$$E_{1,N} \geq c\|u_N\|_{H^{-1}(\Omega)}^2,\ c > 0$$

Multiply the original equation by u_N :

$$\frac{d}{dt}\|u_N\|^2 + \|\Delta u_N\|^2 \leq 2c_0\|\nabla u_N\|^2 + c\|u_N\|^2$$

Combine the two estimates :

$$\frac{dE_{2,N}}{dt} + c(\|u_N\|^2_{H^2(\Omega)} + \|\overline{u}_N\|^4_{L^4(\Omega)} + \langle u_N\rangle^2\|\overline{u}_N\|^2) \leq c',\ c > 0$$

$$E_{2,N} = \delta_1\|u_N\|^2 + E_{1,N}$$

$$E_{2,N} \geq c\|u_N\|^2,\ c > 0$$

Equivalent equations :

$$\frac{\partial u_N}{\partial t} + \chi_{\Omega\setminus D}(x)u_N = \Delta\mu_N$$

$$\mu_N = -\Delta u_N + f_N(u_N)$$

$$\frac{\partial u_N}{\partial \nu} = \frac{\partial \mu_N}{\partial \nu} = 0 \text{ on } \Gamma$$

Multiply the first equation by μ_N and the second by $\frac{\partial u_N}{\partial t}$:

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u_N\|^2 + 2\int_\Omega F_N(u_N)\,dx) + \|\nabla \mu_N\|^2 = -((u_N, \chi_{\Omega\backslash D}(x)\mu_N))$$

Multiply the second equation by $\chi_{\Omega\backslash D}(x)u_N$:

$$((u_N, \chi_{\Omega\backslash D}(x)\mu_N)) = -((\Delta u_N, \chi_{\Omega\backslash D}(x)u_N)) + \int_{\Omega\backslash D} f_N(u_N)u_N\,dx$$

Thus :

$$\frac{d}{dt}(\|\nabla u_N\|^2 + 2\int_\Omega F_N(u_N)\,dx)$$

$$+c(\|\nabla \mu_N\|^2 + \int_{\Omega\backslash D} |f_N(u_N)|\,dx + \int_{\Omega\backslash D} F_N(u_N)\,dx) \leq c'\|u_N\|^2_{H^2(\Omega)} + c'',\ c > 0$$

Combining :

$$\frac{dE_{3,N}}{dt} + c(\|u_N\|^2_{H^2(\Omega)} + \|\overline{u}_N\|^4_{L^4(\Omega)} + \langle u_N\rangle^2\|\overline{u}_N\|^2$$

$$+ \int_{\Omega\backslash D} |f_N(u_N)|\,dx + \int_{\Omega\backslash D} F_N(u_N)\,dx + \|\nabla\mu_N\|^2) \leq c',\ c > 0$$

$$E_{3,N} = \delta_2(\|\nabla u_N\|^2 + 2\int_\Omega F_N(u_N)\,dx) + E_{2,N}$$

$$E_{3,N} \geq c\|u_N\|^2_{H^1(\Omega)} - c',\ c > 0$$

Equivalent equations :

$$(-\Delta)^{-1}\frac{\partial\overline{u}_N}{\partial t} + (-\Delta)^{-1}(\chi_{\Omega\backslash D}(x)u_N - \langle\chi_{\Omega\backslash D}(x)u_N\rangle) = -(\mu_N - \langle\mu_N\rangle)$$

$$\mu_N - \langle\mu_N\rangle = -\Delta\overline{u}_N + f_N(u_N) - \langle f_N(u_N)\rangle$$

Thus :

$$\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1} \leq c(\|u_N\| + \|\nabla \mu_N\|)$$

and

$$\|\frac{\partial u_N}{\partial t}\|_{H^{-1}(\Omega)} \leq c(\|u_N\| + \|\nabla \mu_N\|)$$

Furthermore :

$$\|f_N(u_N) - \langle f_N(u_N)\rangle\| \leq c(\|u_N\|_{H^2(\Omega)} + \|\nabla \mu_N\|)$$

Finally :

$$\begin{aligned}&\tfrac{dE_{3,N}}{dt}+c(\|u_N\|^2_{H^2(\Omega)}+\|\overline{u}_N\|^4_{L^4(\Omega)}+\langle u_N\rangle^2+\|\overline{u}_N\|^2\\&+\|\tfrac{\partial u_N}{\partial t}\|^2_{-1}+\|f_N(u_N)-\langle f_N(u_N)\rangle\|^2\\&+\int_{\Omega\setminus D}|f_N(u_N)|\,dx+\int_{\Omega\setminus D}F_N(u_N)\,dx)\le c',\ c>0\end{aligned}$$

$$\overline{u}_N=u_N-\langle u_N\rangle$$

$$E_{3,N}=\|u_N\|^2_{-1}+\delta_1\|u_N\|^2+\delta_2(\|\nabla u_N\|^2+2\int_\Omega F_N(u_N)\,dx)$$
$\delta_1,\ \delta_2>0$ small

$$E_{3,N}\ge c\|u_N\|^2_{H^1(\Omega)}-c',\ c>0$$

Not sufficient to pass to the limit

We need to estimate $|\langle f_N(u_N)\rangle|$ (hence an estimate on the L^2-norm of $f_N(u_N)$)

To do so, we need an estimate of the form $|\langle u_N\rangle| \leq 1-\delta$, $\delta \in (0,1)$ independent of N

We can prove this only locally in time

We assume that $|\langle u_0\rangle| \leq 1-2\delta$, $\delta > 0$ given

We have

$$\frac{d\langle u_N\rangle}{dt} + c\langle u_N\rangle = -\frac{1}{\mathrm{Vol}(\Omega)}\int_{\Omega\setminus D} \overline{u}_N \, dx$$

This yields

$$\langle u_N(t)\rangle = e^{-ct}\langle u_0\rangle - e^{-ct}\int_0^t e^{cs}\, ds \int_{\Omega\setminus D} \overline{u}_N \, dx$$

and

$$\begin{aligned} &|\langle u_N(t)\rangle| \leq |\langle u_0\rangle| + ce^{-ct}\textstyle\int_0^t e^{cs}\|u_N\|\, ds \\ &\leq 1 - 2\delta + c'(1 - e^{-ct}) \end{aligned}$$

$\rightarrow$ There exists $T_0 = T_0(u_0, \delta) > 0$ independent of N such that

$$|\langle u_N(t)\rangle| \leq 1 - \delta, \; t \in [0, T_0]$$

We can then prove that

$$|\langle f_N(u_N)\rangle| \leq c_\delta \|\overline{u}_N\|_{L^2} \|f_N(u_N) - \langle f_N(u_N)\rangle\|_{L^2} + c'_\delta$$

$\rightarrow$ Uniform estimate on the L^2-norm of $f_N(u_N)$ on $[0, T_0]$

This allows to pass to the limit on $[0, T_0]$

Existence of a local (in time) solution

Remark : We rewrite the equation in the form

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + u - \chi_D(x) u = 0$$

We then have

$$\frac{d\langle u\rangle}{dt} + \langle u\rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_D u\,dx$$

and

$$|\langle u(t)\rangle| \leq e^{-t}|\langle u_0\rangle| + \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)} e^{-t} \int_0^t e^s\,ds$$

This yields

$$|\langle u(t)\rangle| \leq \max(|\langle u_0\rangle|, \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(\Omega)})$$

whence

$$|\langle u(t)\rangle| \leq 1 - \delta$$

where $\delta = \delta(u_0) \in (0, 1)$ is independent of time

$\rightarrow$ The solutions are global in time

Numerical simulations :

One-step algorithm with threshold

The convergence time is faster

The results are better with large inpainting domains

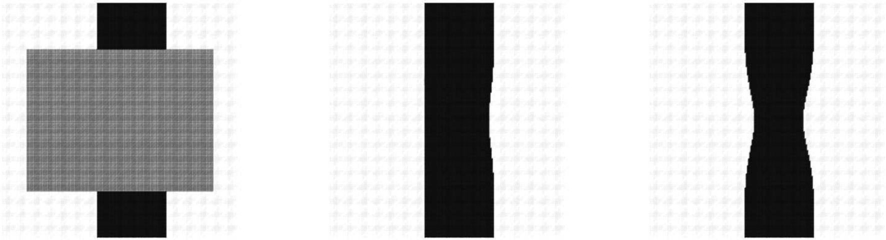


FIGURE – 1. Large inpainting domain 2. Inpainted solution with a logarithmic nonlinearity 3. With a polynomial nonlinearity

Extensions of the model :

Cahn-Hilliard inpainting for multicolor images (L. Cherfils-H. Fakih-A. Miranville, JMIV) :

Multiphasic Cahn-Hilliard system

Each phase corresponds to a color

We consider the hyperplane

$$S = \{c \in R^n \text{ such that } \sum_{i=1}^{n} c_i = 1\}$$

$h = (h_1, ..., h_n) \in S$: damaged image, known on $\Omega \backslash D$

We look for $u = (u_1, ..., u_n) \in S$ such that

$$\begin{aligned}
&\frac{\partial u_i}{\partial t} = \Delta \mu_i + \lambda_0 \chi_{\Omega \setminus D}(x)(h_i - u_i), \quad i = 1, ..., n \\
&\mu_i = f_i(u) - \epsilon^2 \Delta u_i, \quad i = 1, ..., n \\
&\frac{\partial u_i}{\partial \nu} = \frac{\partial \mu_i}{\partial \nu} = 0 \text{ on } \Gamma, \quad i = 1, ..., n \\
&u_i\big|_{t=0} = u_{i,0}, \quad i = 1, ..., n
\end{aligned}$$

$$\begin{aligned}
&f_i(u) = \frac{\partial F(u)}{\partial u_i} - \frac{1}{n} \sum_{j=1}^{n} \frac{\partial F(u)}{\partial u_j}, \quad i = 1, ..., n \\
&F(u) = \frac{1}{n} \sum_{i=1}^{n} u_i^2 (1 - u_i^2)
\end{aligned}$$

Lagrange multiplier to ensure $u \in S$

Well-posedness and regularity of the solutions

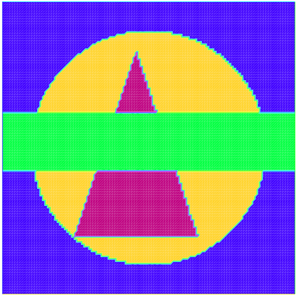
Existence of finite-dimensional attractors

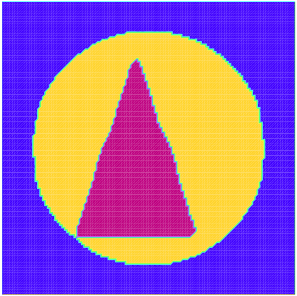
The model is algebraically consistent with the diphasic model

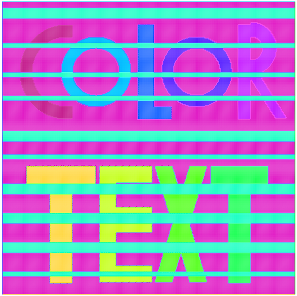
Numerical simulations :

One-step algorithm with threshold

Drawback : not efficient when the number of colors n is large



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Grayscale Cahn-Hilliard inpainting : (L. Cherfils-H. Fakih-A. Miranville, MMS)

Aim : propose a simple model

Known models : heavy to implement numerically

J. Bosch et al. : multiphasic Cahn-Hilliard system ; n : number of shades of gray (not efficient when n is large)

Other models : total variation in H^{-1}, Low Curvature Image Simplifier (similar drawbacks)

Idea : consider a complex version of the Bertozzi et al. model (H. Grossauer-O. Sherzer : complex Allen-Cahn equation)

$h_1 \in L^2(\Omega)$: damaged image ($h_1 : \Omega \to [-1,1]$)

We introduce $h : \Omega \to C$ defined by

$$h = h_1 + ih_2,\ h_2(x) = \sqrt{1 - h_1(x)^2}$$

$\to h \in L^2(\Omega; C)$, $|h| = 1$

Complex version of the Bertozzi et al. model :

$$\begin{array}{l} \frac{\partial u}{\partial t} + \epsilon\Delta^2 u - \frac{1}{\epsilon}\Delta f(u) + \lambda_0 \chi_{\Omega\setminus D}(x)(u-h) = 0 \\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma \\ u\big|_{t=0} = u_0 \end{array}$$

$f(z) = |z|^2 z - z,\ z \in C$

Well-posedness and regularity of the solutions

Existence of finite-dimensional attractors

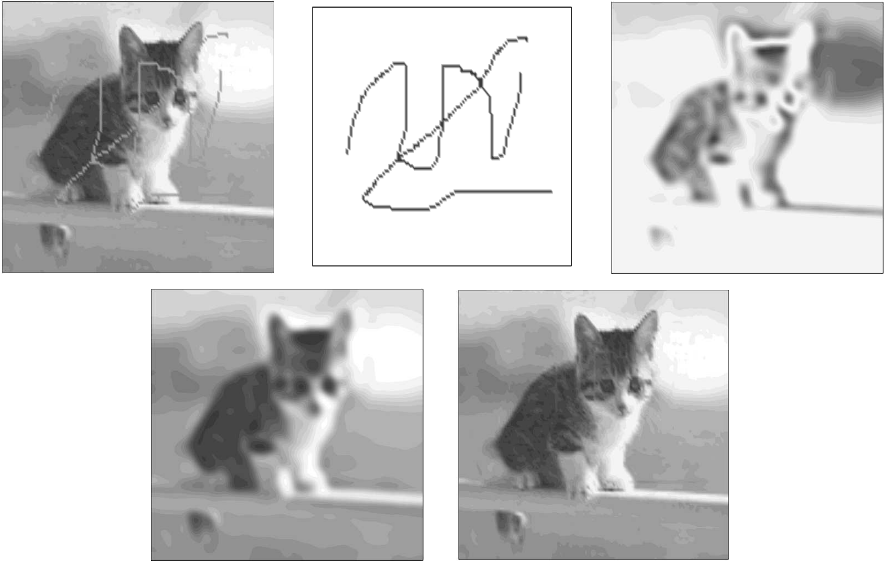
Numerical simulations :

Two-steps algorithm

We use the information on the image known outside the inpainting region

Inpainting result : real part of the solution

We only need to compute two functions whatever the number of shades of gray is

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