Uniqueness and regularity of a diffuse interface problem for binary fluid flows

Andrea Giorgini
Indiana University

NSF-CBMS Conference
Cahn-Hilliard Equation: Recent Advances and Applications

Overview

- **Physical phenomenon:**

  Evolution and mixing of two incompressible and viscous fluids

- **Lecture based on:**

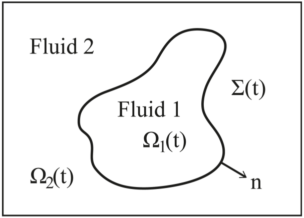
  A. G., A. Miranville & R. Temam,
  Uniqueness and regularity for the Navier-Stokes-Cahn-Hilliard system, SIAM Journal of Mathematical Analysis, to appear

A Sharp Interface model

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \operatorname{div}\big(-pI + 2\nu_i D\boldsymbol{u}\big) = 0 & \text{in } \Omega_i(t) \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega_i(t) \end{cases}$$

with

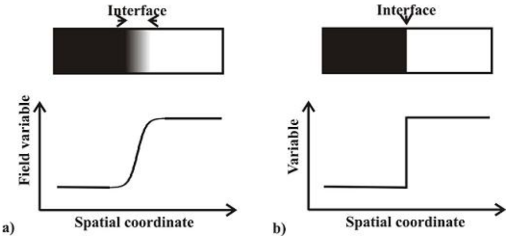
$$\begin{cases} [\boldsymbol{u}]_\Sigma = 0 & \text{on } \Sigma(t) \\ \boldsymbol{u}\cdot\boldsymbol{n} = V & \text{on } \Sigma(t) \\ [-pI + \nu_i D\boldsymbol{u}]_\Sigma \cdot \boldsymbol{n} = \sigma\kappa\boldsymbol{n} & \text{on } \Sigma(t) \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot) & \text{in } \Omega \end{cases}$$



- Local solutions until $\Sigma(t)$ does change its topology (or lose regularity)

Diffuse vs Sharp Interface Modeling

- The interface is a narrow zone with **finite thickness**
- The fluid concentration varies steeply but **continuously** across the interface
- The evolution of the concentration is ruled by the **Cahn-Hilliard equation**



1. Interface: from Lagrangian to Eulerian description
2. Thermodynamic consistent models $\rightarrow$ large deformations of the interface
3. Free boundary problems are recovered in the sharp interface limit

From theory of mixtures...

• Consider two incompressible fluids with densities $\overline{\rho}_1$ and $\overline{\rho}_2$, kinematic viscosities ν_1, ν_2 and partial velocities $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$

• For the mixture, $\rho = \rho_1 + \rho_2$ satisfies

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0$$

where the mass-averaged velocity

$$\boldsymbol{u} = \frac{\rho_1}{\rho}\boldsymbol{u}_1 + \frac{\rho_2}{\rho}\boldsymbol{u}_2$$

• Diffusional continuity equations:

$$\partial_t \rho_j + \operatorname{div}(\rho_j \boldsymbol{u}) = \operatorname{div} \mathbf{J}_i, \quad \text{with} \quad \mathbf{J}_1 + \mathbf{J}_2 = 0$$

• We define $c_j = \frac{\rho_j}{\rho}$ = concentration of fluid $j = 1, 2$ ($c_1 + c_2$=1) and

$$\varphi = c_1 - c_2$$

which satisfies

$$\rho\partial_t\varphi + \rho\boldsymbol{u}\cdot\nabla\varphi = \mathrm{div}\mathbf{J}$$

where by Fick's law

$$\mathbf{J} = 2\mathbf{J}_1 = \nabla\mu$$

In this study we will assume that the two fluids are incompressible with equal densities $\overline{\rho}_1 \approx \overline{\rho}_2 \approx \rho = 1$ (but $\rho_1, \rho_2 \neq \overline{\rho}_1; \overline{\rho}_2$ in the mixture). Then, for the mixture

$$\mathrm{div}\,\boldsymbol{u} = 0$$

In general, $\rho = \rho(\overline{\rho}_1, \overline{\rho}_2, \varphi)$, such as

$$\frac{1}{\rho} = \frac{1+\varphi}{2\overline{\rho}_1} + \frac{1-\varphi}{2\overline{\rho}_2}$$

J. Lowengrub & L. Truskinovsky, Proc. R. Soc. Lond. A 1998

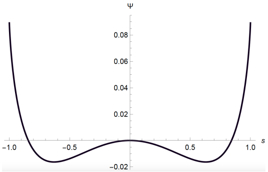
...to Cahn-Hilliard equation

$$\varphi = c_1 - c_2 = \text{difference of concentrations } (\varphi \in [-1, 1])$$

$$\partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi = \Delta \mu, \quad \mu = -\varepsilon \Delta \varphi + \tfrac{1}{\varepsilon} \Psi'(\varphi)$$

$$\mu = \text{chemical potential} = \frac{\delta \mathcal{E}_0}{\delta \varphi}$$

$$\mathcal{E}_0(\varphi) = \int_\Omega \left[\frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} \Psi(\varphi) \right] \mathrm{d}x$$



where

$$\Psi(s) = \frac{\theta}{2} \Big[(1+s) \log(1+s) + (1-s) \log(1-s) \Big] - \frac{\theta_0}{2} s^2, \quad \forall s \in [-1, 1]$$

J.W. Cahn & J.E. Hilliard, J. Chem. Phys. 1958

Navier-Stokes-Cahn-Hilliard system

In a smooth bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$

$\boldsymbol{u}$ = averaged velocity φ = difference of fluid concentrations

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \operatorname{div}(\nu(\varphi) D\boldsymbol{u}) + \nabla \pi = -\operatorname{div}(\nabla\varphi \otimes \nabla\varphi) \\ \operatorname{div} \boldsymbol{u} = 0 \\ \partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi = \Delta \mu \\ \mu = -\Delta\varphi + \Psi'(\varphi) \end{cases}$$

with boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = \boldsymbol{0}, \quad \partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}\mu = 0 & \text{on } \partial\Omega \times (0, T) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \varphi(0) = \varphi_0 & \text{in } \Omega \end{cases}$$

Here $D\boldsymbol{u} = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^t\right)$, $\varepsilon = 1$

Viscosity and Potential

- The viscosity of the mixture ν

$$\nu(s) = \nu_1 \frac{1+s}{2} + \nu_2 \frac{1-s}{2}, \quad \forall s \in [-1, 1]$$

where $\nu_1, \nu_2 > 0$ are the viscosities of the two fluids $(0 < \nu_* \leq \nu(s))$

- The physically relevant free-energy density Ψ is

$$\Psi(s) = \frac{\theta}{2}\Big[(1+s)\log(1+s) + (1-s)\log(1-s)\Big] - \frac{\theta_0}{2}s^2, \quad \forall\, s \in [-1, 1]$$

where θ, θ_0 are constants such that $0 < \theta < \theta_0$

A short literature

- **Constant density $\rho \equiv 1$:**
  - F. Boyer, AA 1999
  - H. Abels, ARMA 2009
  - C. Gal & M. Grasselli, AIHP 2010
  - C. Gal, M. Grasselli & A. Miranville, CVPDE 2016
- **Non-constant density $\rho = \rho(\varphi)$:**
  - H. Abels & E. Feireisl, IUMJ 2008
  - H. Abels, CMP 2009 & SIMA 2012
  - H. Abels, D. Depner & H. Garcke, AIHP 2013 & JMFM 2013
  - C. Gal, M. Grasselli, H. Wu, ARMA 2019

Basic features

- **Energy equation**

$$\mathcal{E}(\boldsymbol{u},\varphi) = \int_\Omega \Big(\frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{2}|\nabla\varphi|^2 + \Psi(\varphi)\Big)\,\mathrm{d}x$$

$$\Downarrow$$

$$\mathcal{E}(\boldsymbol{u}(t),\varphi(t)) + \int_0^t\int_\Omega \Big(|\nabla\mu|^2 + \nu(\varphi)|D\boldsymbol{u}|^2\Big)\,\mathrm{d}x\mathrm{d}s = \mathcal{E}(\boldsymbol{u}_0,\varphi_0) \quad \forall\, t\geq 0$$

- **Conservation of mass**

$$\overline{\varphi}(t) = \frac{1}{|\Omega|}\int_\Omega \varphi(t)\,\mathrm{d}x = \overline{\varphi}_0, \quad \forall\, t\geq 0$$

- **Physical solutions**

$$\varphi\in L^\infty(\Omega\times(0,T)), \quad |\varphi(x,t)|<1,\ a.e.\,(x,t)\in\Omega\times(0,T)$$

Analytical difficulties

- **Navier-Stokes eqs.:**
  - Theory in $d = 3$ dimensions
  - Non-constant viscosity: $-\mathrm{div}(\nu(\varphi)D\boldsymbol{u})$
  - Coupling term: $\mathrm{div}(\nabla\varphi \otimes \nabla\varphi)$

- **Cahn-Hilliard eq.:** The convex part of the potential Ψ is

$$F(s) = \frac{\theta}{2}\Big[(1+s)\log(1+s) + (1-s)\log(1-s)\Big]$$

If we look at its derivatives...

$$F'(s) = \frac{\theta}{2}\log\left(\frac{1+s}{1-s}\right), \quad F''(s) = \frac{\theta}{1-s^2}, \quad F'''(s) = \frac{2\theta s}{(1-s^2)^2}$$

Relative growth conditions

$$F''(s) \leq C\mathrm{e}^{C|F'(s)|}, \quad |F'''(s)| \leq CF''(s)^2$$

Known results

H. Abels, ARMA 2009:

(1.) Global existence of weak solutions (energy space) if $d = 2, 3$:

$$\boldsymbol{u} \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$$
$$\varphi \in L^\infty(0,T;H^1 \cap L^\infty) \cap L^2(0,T;W^{2,p})$$

where $p = 6$ if $d = 3$ and for any $p \in (1,\infty)$ if $d = 2$

(2.) Global strong solutions if $d = 2$ with $\varphi_0 \in H^2$ such that $\mu_0 \in H^1$, and $\boldsymbol{u}_0 \in V_\sigma^{1+s}$ for $s > 0$, where $V_\sigma^{1+s} = (H^1_\sigma, H^2_\sigma)_{s,2}$:

$$\boldsymbol{u} \in L^2(0,T;H^{2+s}) \cap L^\infty(0,T;H^{1+s-\varepsilon}), \quad \varphi \in L^\infty(0,T;W^{2,p})$$

for all $T > 0$, and for any $\varepsilon > 0$ and $2 \le p < \infty$

(3.) Local strong solutions if $d = 3$ with $\varphi_0 \in H^2$ such that $\mu_0 \in H^1$, and $\boldsymbol{u}_0 \in V_\sigma^{1+s}$ for $s > \frac{1}{2}$:

$$\boldsymbol{u} \in L^2(0,T_0;H^{2+s}) \cap L^\infty(0,T_0;H^{1+s-\varepsilon}), \quad \varphi \in L^\infty(0,\infty;W^{2,6})$$

for some $T_0 > 0$ and for all $\varepsilon > 0$.

Results

Theorem (G., Miranville & Temam, 2018)

(1.) *Let* $d = 2$ *and* $\mathcal{E}(\boldsymbol{u}_0, \varphi_0) < \infty$. *Any weak solution satisfies* $\varphi \in L^4(0, T; H^2)$, *is unique and depends continuously from the initial data.*

(2.) *Let* $d = 2$ *and* $\varphi_0 \in H^2$ *such that* $\mu_0 \in H^1$, *and* $\boldsymbol{u}_0 \in H^1_\sigma$. *There exists a global unique strong solution such that*

$$\boldsymbol{u} \in L^2(0, T; H^2) \cap L^\infty(0, T; H^1), \quad \varphi \in L^\infty(0, T; W^{2,p})$$

for all $T > 0$, *and for any* $2 \le p < \infty$.

(3.) *Let* $d = 3$ *and* $\varphi_0 \in H^2$ *such that* $\mu_0 \in H^1$, *and* $\boldsymbol{u}_0 \in H^1_\sigma$. *There exists a local unique strong solution such that, for some* $T_0 > 0$,

$$\boldsymbol{u} \in L^2(0, T_0; H^2) \cap L^\infty(0, T_0; H^1), \quad \varphi \in L^\infty(0, T_0; W^{2,6}).$$

Uniqueness: The classical strategy fails

Given two weak solutions, let us define $(\boldsymbol{u}, \varphi) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, \varphi_1 - \varphi_2)$.

Testing the NS equations by $\boldsymbol{u}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{L^2}^2 + \nu_*\|\nabla \boldsymbol{u}\|_{L^2}^2 + (\nu(\varphi_1) - \nu(\varphi_2)D\boldsymbol{u}_1, \nabla \boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}_1, \boldsymbol{u}) \\ \leq (\nabla\varphi_1 \otimes \nabla\varphi, \nabla \boldsymbol{u}) + (\nabla\varphi \otimes \nabla\varphi_2, \nabla \boldsymbol{u})$$

Testing the CH equation by $(-\Delta_N)^{-1}\varphi$, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_{-1}^2 + \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 \leq C\big(1 + \|\boldsymbol{u}_1\|_{L^3}^2\big)\|\varphi\|_{-1}^2 + \|\boldsymbol{u}\|_{L^2}\|\varphi\|_{-1}$$

where $\|\varphi\|_{-1} = \|\nabla(-\Delta_N)^{-1}\varphi\|_{L^2}$

Uniqueness: idea of the proof

Given two weak solutions, let us define $(\boldsymbol{u}, \varphi) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, \varphi_1 - \varphi_2)$.

Testing the CH eqn by $(-\Delta_N)^{-1}\varphi$, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_{-1}^2 + \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 \le C\big(1 + \|\boldsymbol{u}_1\|_{L^3}^2\big)\|\varphi\|_{-1}^2 + \|\boldsymbol{u}\|_{L^2}\|\varphi\|_{-1}$$

Testing the NS eqns by $A^{-1}\boldsymbol{u}$, where A is the Stokes operator, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_*^2 + (\nu(\varphi_1)D\boldsymbol{u}, \nabla A^{-1}\boldsymbol{u}) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$$

where $\|\boldsymbol{u}\|_* = \|\nabla A^{-1}\boldsymbol{u}\|_{L^2}$ and

$$\mathcal{I}_1 = -((\nu(\varphi_1) - \nu(\varphi_2))D\boldsymbol{u}_2, \nabla A^{-1}\boldsymbol{u})$$
$$\mathcal{I}_2 = (\boldsymbol{u}_1 \otimes \boldsymbol{u}, \nabla A^{-1}\boldsymbol{u}) + (\boldsymbol{u} \otimes \boldsymbol{u}_2, \nabla A^{-1}\boldsymbol{u})$$
$$\mathcal{I}_3 = (\nabla\varphi_1 \otimes \nabla\varphi, \nabla A^{-1}\boldsymbol{u}) + (\nabla\varphi \otimes \nabla\varphi_2, \nabla A^{-1}\boldsymbol{u})$$

Since div $(\nabla \boldsymbol{v})^t = \nabla(\operatorname{div} \boldsymbol{v})$, integrating by parts

$$
\begin{aligned}
(\nu(\varphi_1) D\boldsymbol{u}, \nabla A^{-1}\boldsymbol{u}) &= (\nabla \boldsymbol{u}, \nu(\varphi_1) DA^{-1}\boldsymbol{u}) \\
&= -(\boldsymbol{u}, \nu'(\varphi_1) DA^{-1}\boldsymbol{u} \nabla\varphi_1) - \frac{1}{2}(\boldsymbol{u}, \nu(\varphi_1)\Delta A^{-1}\boldsymbol{u})
\end{aligned}
$$

By the properties of the Stokes operator, there exists $p \in L^2(0,T;H^1)$ such that $-\Delta A^{-1}\boldsymbol{u} + \nabla p = \boldsymbol{u}$ a.e. in $\Omega \times (0,T)$. We have

$$
\|p\|_{L^2} \leq C\|\nabla A^{-1}\boldsymbol{u}\|_{L^2}^{\frac{1}{2}} \|\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}, \quad \|p\|_{H^1} \leq C\|\boldsymbol{u}\|_{L^2}
$$

Therefore, we deduce

$$
\begin{aligned}
-\frac{1}{2}(\boldsymbol{u}, \nu(\varphi_1)\Delta A^{-1}\boldsymbol{u}) &= \frac{1}{2}(\nu(\varphi_1)\boldsymbol{u}, \boldsymbol{u}) - \frac{1}{2}(\nu(\varphi_1)\boldsymbol{u}, \nabla p) \\
&\geq \nu_* \|\boldsymbol{u}\|_{L^2}^2 + \frac{1}{2}(\nu'(\varphi_1)\nabla\varphi_1 \cdot \boldsymbol{u}, p)
\end{aligned}
$$

Setting

$$\mathcal{H} = \frac{1}{2}\|\boldsymbol{u}\|_*^2 + \frac{1}{2}\|\varphi\|_{-1}^2$$

we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} + \nu_*\|\boldsymbol{u}\|_{L^2}^2 + \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 \leq C\mathcal{H} + \sum_{k=1}^{5}\mathcal{I}_k$$

$$\begin{aligned}
\mathcal{I}_1 &= -((\nu(\varphi_1) - \nu(\varphi_2))D\boldsymbol{u}_2, \nabla A^{-1}\boldsymbol{u}) \\
\mathcal{I}_2 &= (\boldsymbol{u}_1 \otimes \boldsymbol{u}, \nabla A^{-1}\boldsymbol{u}) + (\boldsymbol{u} \otimes \boldsymbol{u}_2, \nabla A^{-1}\boldsymbol{u}) \\
\mathcal{I}_3 &= (\nabla\varphi_1 \otimes \nabla\varphi, \nabla A^{-1}\boldsymbol{u}) + (\nabla\varphi \otimes \nabla\varphi_2, \nabla A^{-1}\boldsymbol{u}) \\
\mathcal{I}_4 &= (\boldsymbol{u}, \nu'(\varphi_1)DA^{-1}\boldsymbol{u}\nabla\varphi_1) \\
\mathcal{I}_5 &= -\frac{1}{2}(\nu'(\varphi_1)\nabla\varphi_1 \cdot \boldsymbol{u}, p)
\end{aligned}$$

We control two terms as follows

$$\begin{aligned}\mathcal{I}_5 &\leq C\|\nabla\varphi_1\|_{L^4}\|\boldsymbol{u}\|_{L^2}\|p\|_{L^4}\\ &\leq C\|\varphi_1\|_{H^2}^{\frac{1}{2}}\|\nabla A^{-1}\boldsymbol{u}\|_{L^2}^{\frac{1}{4}}\|\boldsymbol{u}\|_{L^2}^{\frac{7}{4}}\\ &\leq \frac{\nu_*}{8}\|\boldsymbol{u}\|_{L^2}^2 + C\|\varphi_1\|_{H^2}^4\|\boldsymbol{u}\|_*^2\end{aligned}$$

$$\begin{aligned}\mathcal{I}_1 &\leq C\|D\boldsymbol{u}_2\|_{L^2}\|\varphi\|_{L^\infty}\|\nabla A^{-1}\boldsymbol{u}\|_{L^2}\\ &\leq C\|\boldsymbol{u}_2\|_{H^1}\|\nabla\varphi\|_{L^2}\Big[\log\Big(C\frac{\|\varphi\|_{H^2}}{\|\nabla\varphi\|_{L^2}}\Big)\Big]^{\frac{1}{2}}\|\boldsymbol{u}\|_*\end{aligned}$$

We find the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} + \mathcal{G} \le \mathcal{Y}_1\mathcal{H} + \mathcal{Y}_2\Big[\mathcal{H}\mathcal{G}\log\Big(\frac{\mathcal{S}}{\mathcal{G}}\Big)\Big]^{\frac{1}{2}}$$

where

$$\mathcal{G} = \frac{1}{4}\|\nabla\varphi\|_{L^2}^2, \quad \mathcal{Y}_2 = C\|\boldsymbol{u}_2\|_{H^1}, \quad \mathcal{S} = C\|\varphi\|_{H^2}^2$$

$$\mathcal{Y}_1 = C\Big(1 + \|\boldsymbol{u}_1\|_{H^1}^2 + \|\boldsymbol{u}_2\|_{H^1}^2 + \|\nabla\varphi_1\|_{L^\infty}^2 + \|\nabla\varphi_2\|_{L^\infty}^2 + \|\varphi_1\|_{H^2}^4\Big)$$

Since $\mathcal{Y}_2 \in L^2(0,T)$, $\mathcal{S} \in L^1(0,T)$ and $\mathcal{H}(0) = 0$, a result by Li-Titi (Nonlinearity, 2016) entails that $\mathcal{H}(t) = 0$ for all $t \in [0,T]$.

We find the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} + \mathcal{G} \le \mathcal{Y}_1\mathcal{H} + \mathcal{Y}_2\Big[\mathcal{H}\mathcal{G}\log\Big(\frac{\mathcal{S}}{\mathcal{G}}\Big)\Big]^{\frac{1}{2}}$$

where

$$\mathcal{G} = \frac{1}{4}\|\nabla\varphi\|_{L^2}^2, \quad \mathcal{Y}_2 = C\|\boldsymbol{u}_2\|_{H^1}, \quad \mathcal{S} = C\|\varphi\|_{H^2}^2$$
$$\mathcal{Y}_1 = C\Big(1 + \|\boldsymbol{u}_1\|_{H^1}^2 + \|\boldsymbol{u}_2\|_{H^1}^2 + \|\nabla\varphi_1\|_{L^\infty}^2 + \|\nabla\varphi_2\|_{L^\infty}^2 + \|\varphi_1\|_{H^2}^4\Big)$$

Since $\mathcal{Y}_2 \in L^2(0,T)$, $\mathcal{S} \in L^1(0,T)$ and $\mathcal{H}(0) = 0$, a result by Li-Titi (Nonlinearity, 2016) entails that $\mathcal{H}(t) = 0$ for all $t \in [0,T]$.

Remark: This argument shows uniqueness of solutions, but does not provide any control on the distance between two solutions!

Toward a continuous dependence estimate

Logarithmic product estimate if $d = 2$:

$$\|fg\|_{L^2} \leq C\|f\|_{L^2}\|g\|_{H^1}\Big[\log\Big(\mathrm{e}\frac{\|f\|_{H^1}}{\|f\|_{L^2}}\Big)\Big]^{\frac{1}{2}}, \quad f, g \in H^1$$

We improve the bound on $\mathcal{I}_1$

$$\begin{aligned}\mathcal{I}_1 &\leq C\|D\boldsymbol{u}_2\|_{L^2}\|\nabla\varphi\|_{L^2}\Big(\|\nabla A^{-1}\boldsymbol{u}\|_{L^2} + \|\varphi\|_{-1}\Big) \\ &\qquad\qquad \times \Big[\log\Big(C\frac{\|\boldsymbol{u}\|_{L^2} + \|\varphi\|_{L^2}}{\|\nabla A^{-1}\boldsymbol{u}\|_{L^2} + \|\varphi\|_{-1}}\Big)\Big]^{\frac{1}{2}} \\ &\leq \frac{1}{8}\|\nabla\varphi\|_{L^2}^2 + C\|\boldsymbol{u}_2\|_{H^1}^2\Big(\|\boldsymbol{u}\|_*^2 + \|\varphi\|_{-1}^2\Big)\log\Big(\frac{C}{\|\boldsymbol{u}\|_*^2 + \|\varphi\|_{-1}^2}\Big)\end{aligned}$$

We obtain the new differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} \leq \mathcal{Y}_1 \mathcal{H} \log\left(\frac{C}{\mathcal{H}}\right)$$

where

$$\mathcal{H} = \frac{1}{2}\|\boldsymbol{u}\|_*^2 + \frac{1}{2}\|\varphi\|_{-1}^2$$

and

$$\mathcal{Y}_1 = C\Big(1 + \|\boldsymbol{u}_1\|_{H^1}^2 + \|\boldsymbol{u}_2\|_{H^1}^2 + \|\nabla\varphi_1\|_{L^\infty}^2 + \|\nabla\varphi_2\|_{L^\infty}^2 + \|\varphi_1\|_{H^2}^4\Big)$$

Osgood lemma

Lemma

Let f be a measurable function from $[0,T]$ to $[0,a]$, $g \in L^1(0,T)$, and W a continuous and nondecreasing function from $[0,a]$ to $\mathbb{R}^+$. Assume that, for some $c \geq 0$, we have

$$f(t) \leq c + \int_0^t g(s) W(f(s)) \, \mathrm{d}s, \quad \textit{for a.e. } t \in [0,T].$$

- *If $c > 0$, then*

$$-\mathcal{M}(f(t)) + \mathcal{M}(c) \leq \int_0^T g(s) \, \mathrm{d}s, \quad \textit{where} \quad \mathcal{M}(s) = \int_s^a \frac{1}{W(s)} \, \mathrm{d}s.$$

- *If $c = 0$ and $\int_0^a \frac{1}{W(s)} \, \mathrm{d}s = \infty$, then $f(t) = 0$ for a.e. $t \in [0,T]$.*

In our case, $W(s) = s\log(\frac{\mathrm{e}C}{s})$ and $\mathcal{M}(s) = \log(\log(\frac{\mathrm{e}C}{s}))$. Thus, we find

$$-\log\Big(\log\Big(\frac{\mathrm{e}C}{H(t)}\Big)\Big) + \log\Big(\log\Big(\frac{\mathrm{e}C}{H(0)}\Big)\Big) \le \int_0^t \mathcal{Y}_1(s)\,\mathrm{d}s$$

Assuming that

$$\log\Big(\log\Big(\frac{\mathrm{e}C}{H(0)}\Big)\Big) \ge \int_0^t \mathcal{Y}_1(s)\,\mathrm{d}s \quad \text{on } [0, T_0],$$

we deduce that

$$\mathcal{H}(t) \le C\Big(\frac{\mathcal{H}(0)}{C}\Big)^{\mathrm{e}^{-\int_0^t \mathcal{Y}_1(s)\,\mathrm{d}s}} \quad \text{on } [0, T_0]$$

where

$$\mathcal{H}(\cdot) = \frac{1}{2}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_*^2 + \frac{1}{2}\|\varphi_1 - \varphi_2\|_{-1}^2$$

Some remarks...

- Matched viscosities case ($\nu_1 = \nu_2$): we have a stronger continuous dependence estimate:

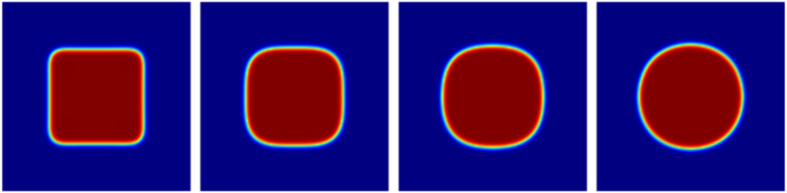
$$\mathcal{H}(t) \leq C\mathcal{H}(0)$$

- Regular potential case (e.g. $\Psi_0 = \frac{1}{4}s^4 - \frac{1}{2}s^2$): the same proof can be adapted to this case, but we cannot improve the exponent

- Strong solutions in three dimensions: the same method provides uniqueness if $\boldsymbol{u}_0 \in H^1_\sigma$

- **Open question**: Weak-strong uniqueness in three dimensions

Surface tension effect



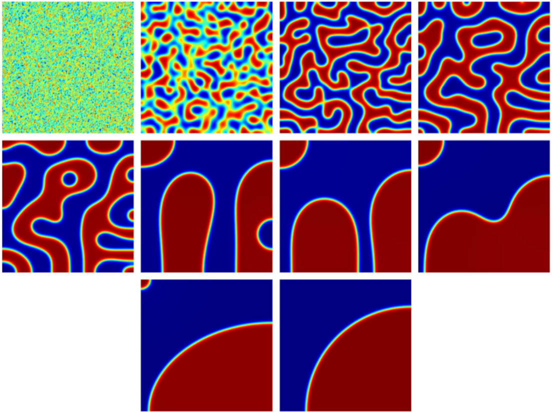
Concentration function at t=0.2, t=1, t=2, t=4

Domain: $\Omega = [-1, 1] \times [-1, 1]$, Parameters: $\nu_1 = \nu_2 = 1$

Initial conditions: $\boldsymbol{u}_0 = 0$, $\varphi_0 =$square shaped fluid bubble

Y. Chen & J. Shen, J. Comput. Phys. 2016

Coarsening dynamics



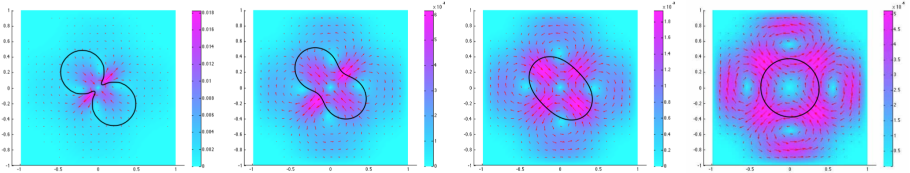
Concentration at t=0, t=0.2, t=0.5, t=1, t=2, t=20, t=50, t=70, t=100, t=500

Domain: $\Omega = [-1, 1] \times [-1, 1]$,   Parameters: $\nu_1 = \nu_2 = 1$

Initial conditions: $\boldsymbol{u}_0 = 0$, φ_0 =random concentration

Y. Chen & J. Shen, J. Comput. Phys. 2016

Coalescence of drops



t=0.01 t=8 t=40 t=260

Level set (black line) $\varphi = 0.5$ (here $\varphi \in [0, 1]$) and velocity field

Domain: $\Omega = [-1, 1] \times [-1, 1]$, Parameters: $\nu_1 = \nu_2 = 1, \varepsilon = 0.01$

Initial conditions: $\boldsymbol{u}_0 = 0$,

$$\varphi_0 = \frac{1}{2}\tanh\left(\frac{-0.2\sqrt{2}+\sqrt{(x+0.2)^2+(y-0.2)^2}}{2\sqrt{2}\varepsilon}\right) + \frac{1}{2}\tanh\left(\frac{-0.2\sqrt{2}+\sqrt{(x-0.2)^2+(y+0.2)^2}}{2\sqrt{2}\varepsilon}\right)$$

Z. Guo, P. Lin & J.S. Lowengrub, J. Comput. Phys. 2014

Thank you for your attention