Doubly nonlocal Cahn-Hilliard equations

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The classical form of the CHE

- Cahn and Hilliard 1958: model for (isothermal) phase separation phenomena in materials made of two components.

$$\partial_t \varphi + \operatorname{div}(M) = 0, \ \mu = -\Delta\varphi + F'(\varphi), \text{ in } \Omega \times (0, \infty).$$

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$$E_{loc}(\varphi) = \int_\Omega \left(\frac{1}{2} |\nabla \varphi|^2 - \theta_c \varphi^2 + \theta F(\varphi) \right) dx,$$

assuming

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- No phase separation when $\theta > \theta_c$ **but only when** $\theta < \theta_c$ (The early stages of the universe)!!!

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- F is a logarithmic (bounded in $\mathbb{R}$) potential

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- Real-world applications:

$$\text{either } m(r) \equiv m_0 > 0 \text{ or } m(r) = m_0\left(1-r^2\right),\ \ r \in [-1,1].$$

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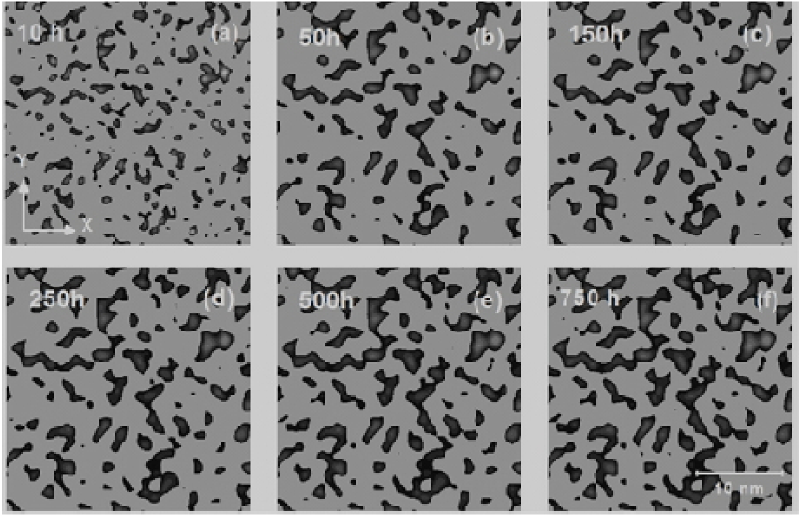

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1. Existence result for a weak (energy) solution: Elliott-Garcke 1996 (F is logarithmic).
2. Uniqueness and regularity still **open** issues!!!

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- Giacomin-Lebowitz 1997 $\Longrightarrow$ nonlocal version of CHE.
- E_{loc} occurs as a first order-approximation of the nonlocal free energy

$$E_{nonloc}(\varphi) = \frac{1}{4}\int_\Omega\int_\Omega J(x-y)\,|\varphi(x)-\varphi(y)|^2 dxdy + \int_\Omega \theta F(\varphi) - \theta_c \varphi^2 dx,$$

where

$$\theta_c := \frac{1}{2}\int_\Omega J(x-y)dy.$$

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- Run simulation of Ising particle model:

https://www.youtube.com/watch?v=kjwKgpQ-I1s

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- We **still** have conservation of mass!

The nonlocal CHE of Giacomin-Lebowitz 1997

Fact

The NCHE $\Longrightarrow$ (second-order) quasi-linear equation:

$$\partial_t \varphi + \nabla \cdot \left(m(\varphi)\, q(x,\varphi) \nabla \varphi + m(\varphi) \nabla a \varphi - m(\varphi) \nabla J * \varphi \right) = 0,$$
$$q(x,\varphi) = a(x) + F''(\varphi).$$

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2. Logarithmic potential F: maximal L^p-regularity and long-term behavior (Giorgini-Gal-Grasselli 2017: **2D results only**) *3D case still open*!!!

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where

$$\begin{aligned} A &= -\Delta_N \text{ (The Neumann Laplacian on } \Omega), \\ B\varphi &= a(x)\varphi - J * \varphi = \int_\Omega J(x-y)\left(\varphi(x) - \varphi(y)\right) dy, \end{aligned}$$

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- Multiscale heterogeneous environment Ω: Neumana-Tartakovsky 2009, Vlahos-Isliker-Kominis-Hizonidis 2008;

anamolous (nonlocal) transport law replaces local one!

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- The classical conservation law $\partial_t \varphi + \mathrm{div}(M) = 0$ must be replaced by a nonlocal formulation for mass transport:

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$$A\mu = \text{P.V.} \int_\Omega K(x-y)\,(\mu(y) - \mu(x))\,dy$$
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- K encodes the physical properties of the environment Ω in a manner in which mass is being transported throughout Ω.

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Part 1: The strong-to-strong interaction case

- Fix $s \in (0,1)$, and set

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$$(-\Delta)^s_\Omega u(x) = \text{P.V. } C_{d,s} \int_\Omega \frac{(u(x) - u(y))}{|x-y|^{d+2s}} dy, \ x \in \Omega, \ u \in \mathcal{L}(\Omega).$$

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- $(-\Delta)^s_{\Omega,D}$ = realization of $(-\Delta)^s_\Omega$ on $L^2(\Omega)$ with the Dirichlet boundary condition. We have

$$\begin{aligned} \text{Dom}((-\Delta)^s_{\Omega,D}) &= \{u \in W^{s,2}_0(\Omega), \ (-\Delta)^s_\Omega u \in L^2(\Omega)\} \\ (-\Delta)^s_{\Omega,D} u &= (-\Delta)^s_\Omega u. \end{aligned}$$

Part 1: The strong-to-strong interaction case

- Here, $W_0^{s,2}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{s,2}}$ with

$$\begin{aligned}\|u\|_{W^{s,2}}^2 &= \frac{C_{d,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dxdy + \int_\Omega |u(x)|^2 \, dx \\ &=: \mathcal{E}_A(u, u) + \|u\|_{L^2(\Omega)}^2 .\end{aligned}$$

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$$\begin{aligned} \|u\|^2_{W^{s,2}} &= \frac{C_{d,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dxdy + \int_\Omega |u(x)|^2 dx \\ &=: \mathcal{E}_A(u,u) + \|u\|^2_{L^2(\Omega)}. \end{aligned}$$

- Note $W_0^{s,2}(\Omega) \subset L^{2q}(\Omega)$ with $q = \frac{d}{d-2s}$ if $d > 2s$ and any $q \in (1, \infty)$ if $d \leq 2s$, and

$$W_0^{s,2}(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega).$$

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- Here, $W_0^{s,2}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{s,2}}$ with

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- Note $W_0^{s,2}(\Omega) \subset L^{2q}(\Omega)$ with
$q = \frac{d}{d-2s}$ if $d > 2s$ and any $q \in (1,\infty)$ if $d \leq 2s$, and

$$W_0^{s,2}(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega).$$

- It can be proven that

$$\mathrm{Dom}((-\Delta)_{\Omega,D}^s) \subset L^\infty(\Omega) \text{ if } s > \frac{d}{4}.$$

Part 1: The strong-to-strong interaction case

- Recall the doubly nonlocal CHE in abstract form reads

$$\partial_t \varphi + A\mu = 0, \;\; \mu = B\varphi + F^{'}(\varphi) \;\text{ in } \Omega \times (0, \infty).$$

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Table:

Model	Classical CHE	Doubly nonlocal CHE, case (4)
A	$-\Delta_{\Omega,N}$	$(-\Delta)^{s_1}_{\Omega,D}$, $s_1 \in (0,1)$
B	$-\Delta_{\Omega,N}$	$(-\Delta)^{s_2}_{\Omega,D}$, $s_2 \in (0,1)$

Table:

Model	CHE: anamolous transport	CHE: nonlocal strong energy
A	$(-\Delta)^{s}_{\Omega,D}$, $s \in (0,1)$	$-\Delta_{\Omega,N}$
B	$-\Delta_{\Omega,N}$	$(-\Delta)^{s}_{\Omega,D}$, $s \in (0,1)$

Part 1: The strong-to-strong interaction case

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- Results:

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- **The elliptic problem**:

$$Bu(x) + f(u(x)) = h(x), \; x \in \Omega,$$

where $h \in L^p(\Omega)$ for some $p > 1$. Here, $f = F^{'} \in C^1(\mathbb{R})$ is a nonlinear function which satisfies

$$f(t)t \geq \alpha_0 t^2 - \alpha_1, \; f^{'}(t) \geq -\alpha_2. \text{ for all } t \in \mathbb{R}, \; |t| \geq t_0,$$

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- We say that u is a bounded generalized solution if $u \in W^{s,2}_0(\Omega) \cap L^\infty(\Omega)$ and

$$\mathcal{E}_B(u, v) + \int_\Omega f(u(x))\, v(x)\, dx = \int_\Omega h(x)\, v(x)\, dx,$$

for all $v \in W^{s,2}_0(\Omega)$.

Part 1: The strong-to-strong interaction case

Theorem

Under the above assumptions on f, there is at least one bounded solution provided that $h \in L^p(\Omega)$ with $p > \frac{d}{2s}$. Moreover, we have

$$\|u\|_{L^\infty(\Omega)} \leq C\left(1 + \|h\|_{L^p(\Omega)}\right),$$

for some constant $C > 0$ independent of u and h.

Corollary

Under the same assumptions, if $h \in L^p(\Omega) \cap L^2(\Omega)$, then $u \in D(B) \cap L^\infty(\Omega)$ such that

$$\|Bu\|_{L^2(\Omega)} \leq Q\left(1 + \|h\|_{L^p(\Omega) \cap L^2(\Omega)}\right),$$

for some function $Q > 0$ independent of u and h.

Strong solutions

- Energy space

$$Z = \left\{ (u_0, \mu_0) \in D(B) \times W_0^{l,2}(\Omega) \right\}, \; D(B) = D((-\Delta)^s_{\Omega,D})$$

with norm (with respect to the pair (u_0, μ_0)),

$$\|u_0\|_Z^2 = \|u_0\|_{D(B)}^2 + \|\mu_0\|_{W^{l,2}}^2 ,$$

where μ_0 is computed via the equation

$$\mu_0 = Bu_0 + f(u_0) \text{ in } \Omega.$$

Strong solutions

Definition

Let $0 < T < +\infty$ be given. We say u is a strong solution if u, μ satisfy

$$u \in L^\infty(0, T; D(B) \cap L^\infty(\Omega)), \ \partial_t u \in L^2(0, T; W_0^{s,2}(\Omega)),$$
$$\mu \in L^\infty(0, T; W_0^{l,2}(\Omega)) \cap L^2(0, T; D(A)).$$

In particular, for the strong solution we have $\partial_t u = -A\mu$, a.e. in $\Omega \times (0, T)$ and $\mu = Bu + f(u)$, a.e. in $\Omega \times (0, T)$.

Strong solutions

1. Regularized version/problem for $(u, \mu) = (u_{\epsilon,\alpha}, \mu_{\epsilon,\alpha})$:

$$\partial_t u = -A\mu, \;\; \mu = \alpha \partial_t u + Bu + f_\varepsilon(u), \qquad ((\mathrm{P}_{\epsilon,\alpha}))$$

where $\left\{ f_\epsilon = F_\epsilon' \right\}$ is such that $f_\epsilon \to f$ uniformly on compact intervals of $\mathbb{R}$, with the property that $\left| f_\epsilon'(s) \right| \leq c_{f,\epsilon}$.

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2. Existence of a strong solution to $(\mathrm{P}_{\epsilon,\alpha})$ by a backward (**finite-difference**) Euler scheme.
3. Derive uniform estimates and pass to the limit as $(\epsilon,\alpha) \to (0,0)$.
4. **Main assumption** on $F \in C^2(\mathbb{R})$, (Hf-1): $\lim_{|s|\to\infty} F(s) = \infty$ and for some $c_F, c_1 > 0$, $c_2 \geq 0$,

$$F^{'}(s)s \geq c_1 s^2 - c_2 \text{ and } f^{'}(s) = F''(s) \geq -c_F, \text{ for all } s \in \mathbb{R}.$$

Strong solutions

Theorem

Let $(u_0, \mu_0) \in Z$ *for some* $s > \frac{d}{4}$. *Then there exists at least one strong solution in the sense of definition.*

Weak solutions

- Weak energy space

$$Y = \left\{ u \in W_0^{s,2}(\Omega) : F(u) \in L^1(\Omega) \right\}$$

with the following metric

$$d(u_1, u_2) = \|u_1 - u_2\|_{W_0^{s,2}} + \left| \int_\Omega F(u_1) - F(u_2)\, dx \right|^{1/2}.$$

- Two more assumptions on F :

(Hf-2) There exists a constant $c_f > 0$ and $p \in (1, 2]$ such that

$$|f(s)|^p \le c_f \left(|F(s)| + 1 \right), \text{ for all } s \in \mathbb{R}.$$

(Hf-3) There exist $C_1 > 0$, $C_2 \ge 0$ and $p \in (1, 2]$ such that

$$F(s) \ge C_1 |s|^{p/(p-1)} - C_2, \quad \text{for all } s \in \mathbb{R}.$$

- $F(s) = \theta s^4 - \theta_c s^2$ satisfies (Hf-1)-(Hf-3) with $p = 4/3$ (and so $\overline{p} := p/(p-1) = 4$).

Weak solutions

- **Weak solution**: $u_0 \in Y$ and u satisfies

$$u \in L^\infty(0,T;Y),\ \partial_t u \in L^2(0,T;W_0^{-l,2}(\Omega)),$$
$$\mu \in L^2(0,T;W_0^{l,2}(\Omega)),$$
$$F(u) \in L^\infty\left(0,T;L^1(\Omega)\right),\ f(u) \in L^\infty(0,T;L^p(\Omega))$$

Definition

- for every $v \in W_0^{s,2}(\Omega) \cap L^{\overline{p}}(\Omega)$, $\omega \in W_0^{l,2}(\Omega)$, a.e. $t \in (0,T)$ we have

$$\langle \partial_t u(t), \omega\rangle + \mathcal{E}_A(\mu(t), \omega) = 0,$$
$$\mathcal{E}_B(u(t), v) + \langle f(u(t)), v\rangle = (\mu(t), v).$$

- We have $u(0) = u_0$ in Ω.

Weak solutions

- Energy identity: test with $\omega = \mu$ and $v = \partial_t u$ in $L^2(\Omega)$ and add the resulting equations:

$$\frac{d}{dt}\left(\mathcal{E}_B(u(t), u(t)) + \int_\Omega F(u(t))\, dx\right) + \mathcal{E}_A(\mu(t), \mu(t)) = 0.$$

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- It is justifiable for strong solutions u_n, with $\left(u_{0n}, \mu_{0n}\right) \in Z$. Approximate $u_0 \in Y$ by a sequence $\left(u_{0n}, \mu_{0n}\right) \in Z$.

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- Use the energy identity to control the solutions uniformly with respect to $n \to \infty$.
- Pass to the limit in n using the uniform (weak) energy bounds.

Weak solutions

Theorem

Let F satisfy the assumptions (Hf-1), (Hf-2), (Hf-3) and assume $s > \frac{d}{4}$. For every initial datum $u_0 \in Y$, there exists at least one weak solution in the sense of the previous definition. Moreover,

$$u \in L^\infty\left(0, T; L^{\overline{p}}\left(\Omega\right)\right), \text{ for any } T > 0.$$

Problem

It may be possible to remove the condition $s > \frac{d}{4}$ (recall that $B = (-\Delta)^s_{\Omega,D}$, $s \in (0,1)$) by using further perturbation arguments!

References: Part 1 (The strong-to-strong interaction case)

C.G. Gal, On the strong-to-strong interaction case for doubly nonlocal Cahn-Hilliard equations. Discrete Contin. Dyn. Syst. 37 (2017), 131–167.

C.G. Gal, Nonlocal Cahn-Hilliard equations with fractional dynamic boundary conditions. European J. of Applied Mathematics (2017), 53 pages, doi: 10.1017/S0956792516000504.

Part 2: The strong-to-weak interaction case

- The doubly nonlocal CHE reads (more generally) as

$$\partial_t \varphi + A\mu = 0, \ \mu = B\varphi + F^{'}(\varphi) \ \text{in } \Omega \times (0, \infty).$$

where

$$\begin{aligned} A\mu &= \text{P.V.} \int_\Omega K(x-y)\left(\mu(y) - \mu(x)\right) dy, \\ B\varphi &= \text{P.V.} \int_\Omega J(x-y)\left(\varphi(y) - \varphi(x)\right) dy. \end{aligned}$$

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- Take for simplicity again $A = (-\Delta)^{l}_{\Omega,D}$, for $l \in (0,1)$ (namely $\rho(r) = C_{d,l} r^{-2l-d}$), **but if** $J \in L^1_{loc}$, we have

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Part 2: The energy identity and estimates

Definition

1. Setting $b(x,\varphi) := a(x)\varphi + F'(\varphi)$, then for every $\psi \in W_0^{l,2}(\Omega)$, a.e. $t \in (0,T)$ we have

$$\langle \varphi_t, \psi \rangle + \mathcal{E}_A(\mu, \psi) = 0,$$
$$\mu = b(x,\varphi) - J * \varphi \text{ a.e. in } \Omega.$$

- Use proper test functions ψ to produce meaningful energy estimates!

Part 2: The energy identity and estimates

Definition

1. Setting $b(x,\varphi) := a(x)\varphi + F'(\varphi)$, then for every $\psi \in W_0^{l,2}(\Omega)$, a.e. $t \in (0,T)$ we have

$$\langle \varphi_t, \psi \rangle + \mathcal{E}_A(\mu, \psi) = 0,$$
$$\mu = b(x,\varphi) - J * \varphi \text{ a.e. in } \Omega.$$

- Use proper test functions ψ to produce meaningful energy estimates!
  1. Choose $\psi = \mu$ and test the second equation by $\partial_t \varphi$:

$$0 = \frac{d}{dt}\left(\frac{1}{4}\int_{\Omega\times\Omega} J(x-y)|\varphi(x)-\varphi(y)|^2 dxdy + \int_\Omega F(\varphi)dx\right) + \mathcal{E}_A(\mu,\mu)$$

where

$$\mathcal{E}_A(\mu,\mu) = \frac{C_{d,l}}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{d+2l}} dxdy.$$

Part 2: The energy identity and estimates

Assumptions on the potential F (Think of $F^{'} = f$ as a polynomial of arbitrary growth).

- There exist $c_1 > 0$, $c_2 \geq 0$ and $p \in (1, 2]$ such that

$$|F'(s)|^p \leq c_1 |F(s)| + c_2, \qquad \forall s \in \mathbb{R}.$$

- Bounds on $(0, T)$ with no sign assumption on J:

$$\begin{array}{ccc} \mu \in L^2(0, T; W_0^{l,2}(\Omega)), & & F(\varphi) \in L^\infty\left(0, T; L^1(\Omega)\right) \\ \Downarrow & & \Downarrow \\ \partial_t \varphi \in L^2(0, T; W_0^{-l,2}(\Omega)) & & ? \end{array}$$

provided that $\varphi_0 \in L^2(\Omega)$ and $F(\varphi_0) \in L^1(\Omega)$.

Part 2: The energy identity and estimates

- Bounds on $(0, T)$:

$$\varphi \overset{?}{\in} L^{\infty}\left(0, T; L^{2}\left(\Omega\right)\right) \cap L^{2}(0, T; W_{0}^{l,2}\left(\Omega\right)).$$

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- **Why care?**

$$L^{2}(0, T; W_{0}^{l,2}\left(\Omega\right)) \cap H^{1}(0, T; W_{0}^{-l,2}\left(\Omega\right)) \overset{c}{\hookrightarrow} L^{2}(0, T; L^{2}\left(\Omega\right)).$$

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- Recall $b(x, \varphi) := a(x)\varphi + F'(\varphi)$, then for every $\psi \in W_{0}^{l,2}\left(\Omega\right)$, a.e. $t \in \left(0, T\right)$ we have

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$$\langle\varphi_t,\psi\rangle+\mathcal{E}_A(\mu,\psi)=0,$$
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- Key point: use test function $\psi=\varphi$. But this requires dealing with **doubly interaction terms** in $\mathcal{E}_A(\mu,\varphi)$!!!

Part 2: The energy identity and estimates

- The energy identity

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2_{L^2(\Omega)} + \mathcal{E}_A(\mu,\varphi) = 0,$$

but $\mu = a(x)\varphi - J*\varphi + F'(\varphi)$ **and**

$$\begin{aligned}\mathcal{E}_A(\mu,\varphi) &= \int_\Omega\int_\Omega \rho(|x-y|)(\mu(x)-\mu(y))(\varphi(x)-\varphi(y))\,dydx \\ &= I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{cases} I_1 := \int_\Omega\int_\Omega \rho(|x-y|)(a(x)+q_F(\varphi))(\varphi(x)-\varphi(y))^2\,dydx, \\ I_2 := \int_\Omega\int_\Omega \rho(|x-y|)(a(x)-a(y))\varphi(y)(\varphi(x)-\varphi(y))\,dydx, \\ I_3 := \int_\Omega\int_\Omega \rho(|x-y|)((J*\varphi)(x)-(J*\varphi)(y))(\varphi(x)-\varphi(y))\,dy \end{cases}$$

and we have set

$$q_F(\varphi) := \frac{F'(\varphi(x)) - F'(\varphi(y))}{\varphi(x)-\varphi(y)}.$$

Part 2: The energy identity and estimates

- Assume that $a(x) + F''(s) \geq c_0$, a.e. $x \in \Omega$, $s \in \mathbb{R}$.
  $\Longrightarrow a(x) + q_F(\varphi) \geq c_0$ and **so**

$$I_1 \geq c_0 \int_\Omega \int_\Omega \rho\,|x-y|\,(\varphi(x) - \varphi(y))^2\,dydx = c_0 \mathcal{E}_A(\varphi, \varphi).$$

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- Thus we have

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2_{L^2(\Omega)} + c_0\|\varphi\|^2_{W_0^{s,2}} \leq |I_2| + |I_3|;$$

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- I_2 can be estimated like I_3. Notice first

$$\begin{aligned} I_3 &= \int_\Omega \int_\Omega \rho(|x-y|)\left((J*\varphi)(x) - (J*\varphi)(y)\right)(\varphi(x) - \varphi(y)) \, dydx \\ &\leq \frac{c_0}{4} \int_\Omega \int_\Omega \rho(|x-y|)(\varphi(x) - \varphi(y))^2 \, dydx \\ &+ \frac{1}{c_0} \int_\Omega \int_\Omega \rho(|x-y|)\left((J*\varphi)(x) - (J*\varphi)(y)\right)^2 dydx. \end{aligned}$$

Part 2: The energy identity and estimates

- We split the last integral into **two parts**:

$$\int_\Omega \int_\Omega \rho\left(|x-y|\right)\left(\left(J*\varphi\right)(x)-\left(J*\varphi\right)(y)\right)^2 dydx = A+B,$$

where

$$\begin{cases} A := \int_\Omega \int_{\Omega:|x-y|\geq 1} \rho\left(|x-y|\right)\left(\left(J*\varphi\right)(x)-\left(J*\varphi\right)(y)\right)^2 dydx, \\ B := \int_\Omega \int_{\Omega:|x-y|<1} \rho\left(|x-y|\right)\left(\left(J*\varphi\right)(x)-\left(J*\varphi\right)(y)\right)^2 dydx. \end{cases}$$

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- Consider $\widetilde{\varphi}_{|\Omega} = \varphi$ and $\widetilde{\varphi}_{|\mathbb{R}^d\setminus\Omega} = 0$. Recall that $\rho(r) = C_{d,l} r^{-d-2l}$, $l \in (0,1)$. We have

$$\begin{aligned} |A| &\leq 2\int_\Omega \int_{\Omega:|x-y|\geq 1} \rho(|x-y|)\left(|(J*\varphi)(x)|^2 + |(J*\varphi)(y)|^2\right) dydx \\ &\leq 2\|\rho\|_{L^\infty[1,\infty)}\left(2|\Omega|\,\|J\|_{L^1}^2\,\|\varphi\|_{L^2}^2\right) \\ &\leq C_\rho |\Omega|\,\|J\|_{L^1}^2\,\|\varphi\|_{L^2}^2 . \end{aligned}$$

Part 2: The energy identity and estimates

1. For the B-term, use the Young convolution theorem in $\mathbb{R}^d$, so

$$
\begin{aligned}
|B| &\leq \int_\Omega \int_{B_1} \rho\left(|z|\right)\left(\left(J*\varphi\right)(x) - \left(J*\varphi\right)(z+x)\right)^2 dzdx \\
&\leq \int_\Omega \int_{B_1} \left(\frac{\left|\left(J*\varphi\right)(x) - \left(J*\varphi\right)(z+x)\right|}{|z|}\right)^2 |z|^2 \rho\left(|z|\right) dzdx \\
&\leq \int_\Omega \int_{B_1} \left(\int_0^1 \left|\nabla J*\varphi\left(x+tz\right)\right| dt\right)^2 |z|^2 \rho\left(|z|\right) dzdx \\
&\leq \int_{\mathbb{R}^d} \int_{B_1} \int_0^1 \left|\nabla J*\widetilde{\varphi}\left(x+tz\right)\right|^2 |z|^2 \rho\left(|z|\right) dtdzdx \\
&\leq \|\nabla J\|^2_{L^1(\mathbb{R}^d)} \int_{B_1} \int_0^1 |z|^2 \rho\left(|z|\right) \|\widetilde{\varphi}\|^2_{L^2(\mathbb{R}^d)} dtdz \\
&\leq |S_{d-1}| \, \|J\|^2_{W^{1,1}(\mathbb{R}^d)} \left(\int_0^1 r^{d+1}\rho\left(r\right) dr\right) \|\varphi\|^2_{L^2(\Omega)} \\
&= \overline{C}_\rho \, |S_{d-1}| \, \|J\|^2_{W^{1,1}(\mathbb{R}^d)} \, \|\varphi\|^2_{L^2(\Omega)} .
\end{aligned}
$$

- **Key assumption**: $J \in W^{1,1}_{loc}\left(\mathbb{R}^d\right)$. We have derived

$$\frac{d}{dt}\left\|\varphi\right\|^2_{L^2(\Omega)} + c_0\left\|\varphi\right\|^2_{W_0^{s,2}} \leq C\left\|\varphi\right\|^2_{L^2(\Omega)}.$$

which implies

$$\varphi \in L^\infty\left(0, T; L^2\left(\Omega\right)\right), \ \varphi \in L^2(0, T; W_0^{l,2}\left(\Omega\right)).$$

Part 2: The strong-to-weak interaction case

Theorem

Let $\varphi_0 \in L^2(\Omega)$ such that $F(\varphi_0) \in L^1(\Omega)$ and suppose the previous assumptions on F, J. Then, for every $T > 0$ there exists a (unique) weak solution φ satisfying the weak formulation. Furthermore, the following energy identity holds for any $t \geq 0$,

$$\mathcal{N}(\varphi(t)) + \int_0^t \|\mu(\tau)\|^2_{W_0^{s,2}} d\tau = \mathcal{N}(\varphi_0)$$

and the functions $t \mapsto \|\varphi(t)\|^2_{L^2(\Omega)}$ and $t \mapsto (F(\varphi(t)), 1)_{L^2(\Omega)}$ are absolutely continuous on $[0, T]$. Here, we have

$$\mathcal{N}(\varphi) = \frac{1}{4}\int_{\Omega\times\Omega} J(x-y)|\varphi(x)-\varphi(y)|^2 dxdy + \int_\Omega F(\varphi)dx.$$

Part 2: The strong-to-weak interaction case, further results

1. Regularity of weak solutions: existence of strong solutions! $\Longrightarrow$ finite dimensional global attractors!

C.G. Gal, Doubly nonlocal Cahn-Hilliard equations. Annales Henry Poincare Nonlin. Anal. (2017), to appear.

Part 2: The strong-to-weak interaction case, further results

1. Regularity of weak solutions: existence of strong solutions! $\Longrightarrow$ finite dimensional global attractors!
2. Convergence to a single steady state:

$$\varphi(t) \to \varphi_* \text{ in } L^\infty(\Omega)\text{-topology}$$

where

$$a(x)\varphi_* - J * \varphi_* + F'(\varphi_*) = const. \text{ in } \Omega.$$

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Many open questions

- The *weak-to-weak* interaction case when both $\rho, J \in L^1_{loc}\left(\mathbb{R}^d\right)$.
- The *weak-to-strong* interaction case when $\rho \in L^1_{loc}\left(\mathbb{R}^d\right)$ and $J \notin L^1_{loc}\left(\mathbb{R}^d\right)$.
- F is a singular (log) potential in **all** cases of interaction.
- Further regularity of weak solutions: $\varphi \in C^{\beta/2,\beta}((0,T) \times \overline{\Omega})$?
- Either one of the nonnegative kernels K, J is *not* radially symmetric. What happens?
- The case $\Omega = R^d$.
- Numerical treatment? None but fundamental issue!

Any questions???