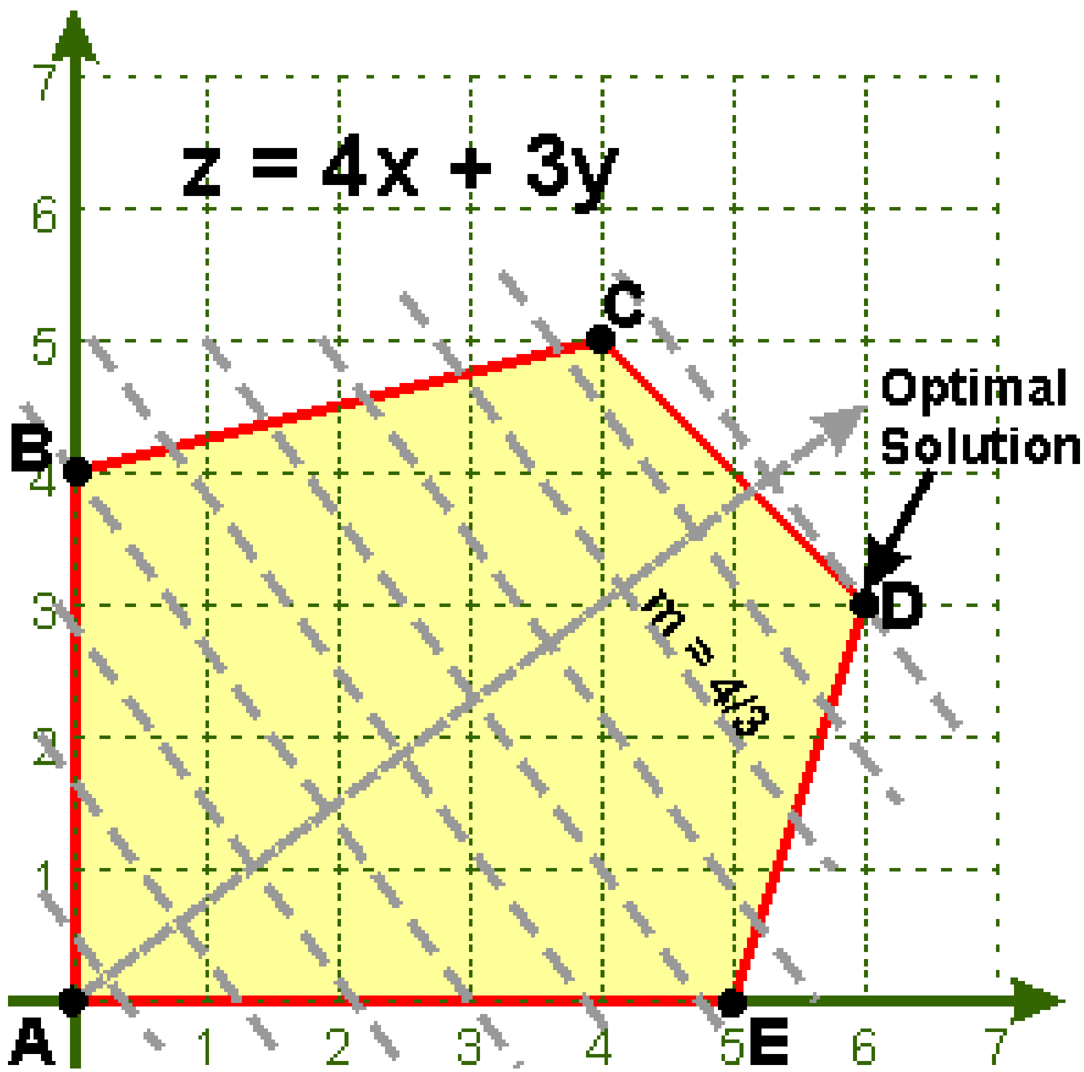
# Review of Linear Programming (LP), Integer Programming (IP), and Duality




Prof. Santosh Kumar
Dept. of Computer Science
University of Memphis
Fall 2008

# An Example

- Giapetto produces two types of toys
- Can sell at most 40 soldiers per week
- Can use at most 100 finishing hours
- Can use at most 80 carpentry hours
- Want to maximize the profits

| **Toy** | **Sale Price** | **Material Cost** | **Labor cost** | **Finishing Hours** | **Carpentry Hours** |
|---|---|---|---|---|---|
| Soldier | $27 | $10 | $14 | 2 | 1 |
| Train | $21 | $9 | $10 | 1 | 1 |

# Derive a Mathematical Model

- Decision Variables
  - What key parameter to vary?
  - $x_1$ : Number of soldiers produced per week
  - $x_2$ : Number of trains produced per week
- Objective Function
  - How to cast the real objective as a function of the decision variables?
- Constraints
  - What limits the decision variables from taking on arbitrary values?
  - Any signs on the decision variables?

# Assumptions for an LP Model

- Linearity
  - The objective function should be linear in decision variables.
  - The constraints should be linear, as well.
- Divisibility
  - The decision variables can take on fractional values (from the real domain)
- Certainty
  - All coefficients are deterministic

# Feasible Region and Optimality

- Feasible Region
  - Set of all points (tuple consisting of a value for each decision variable) that satisfy all LP constraints and the sign restrictions
- An Optimal Solution
  - For max problem, a point in the feasible region with the largest objective function value
- Are Optimal Solutions necessarily unique?

# Graphical Solution to an LP

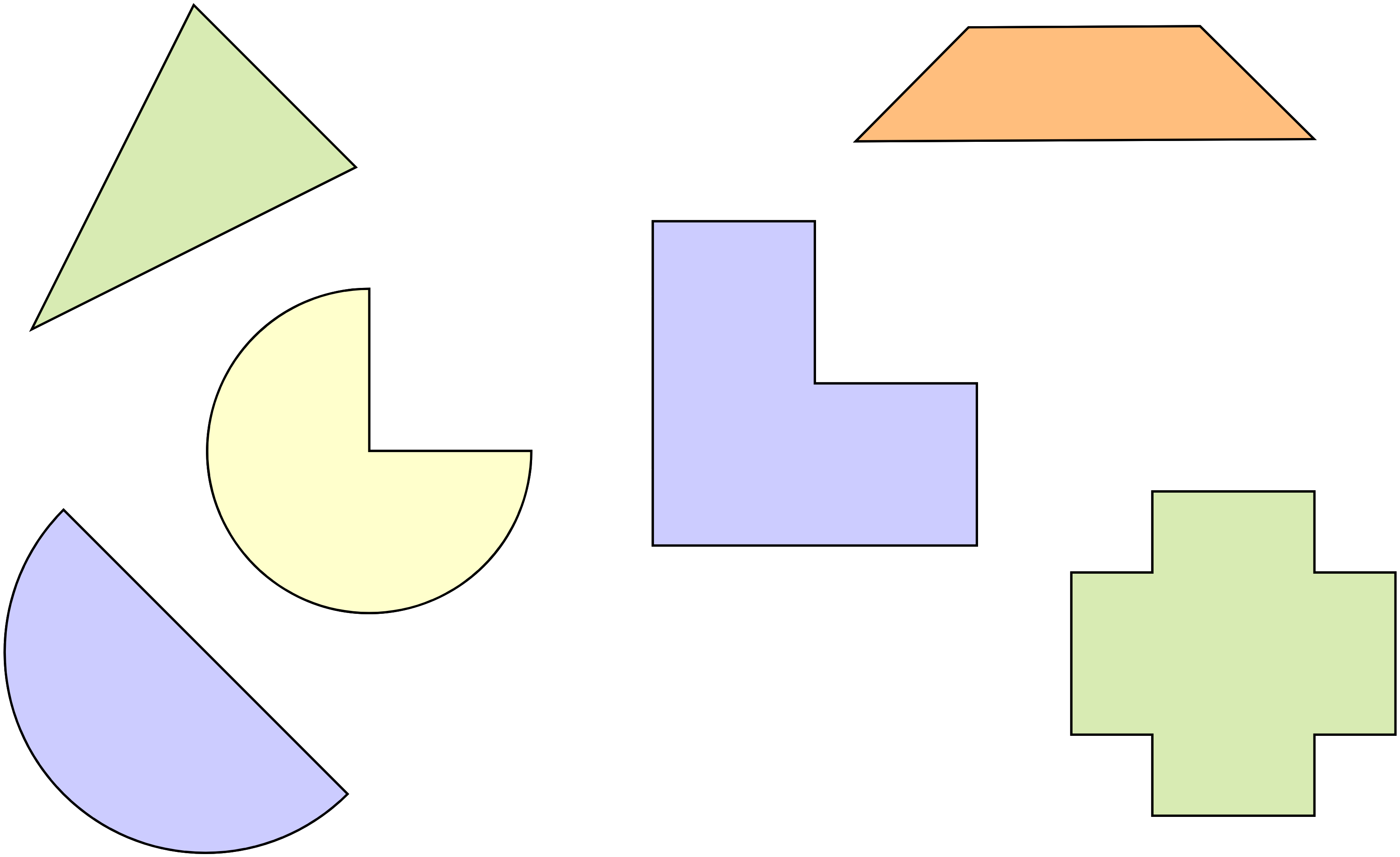
# Classifying the Constraints

- Binding Constraints
  - Equality holds at the optimal point
- Nonbinding Constraints
  - Equality does not hold at the optimal point

# Some Definitions

- **Convex Set**
  - A set *S* of real points such that the line segments joining any pair of points in *S* completely belongs to *S*.
- **Extreme Point** (or **corner points**)
  - For any convex set *S*, a point *P* is an **extreme point** if each line segment that lies completely in *S* and contains the point *P* has *P* as an end point of the line segment.

# Examples



# Role of Convexity in LP

- Feasible region of any LP is a convex set
- Number of extreme points is finite
- If an LP has an optimal solution, it has an extreme point that is optimal
  - Can focus our search for optimality on just the extreme points as opposed to all points in the feasible region

# Special Cases

- Alternative or multiple optimal solutions
  - Can use secondary criteria to make a decision
    - Goal programming is often used in these cases
- Infeasible LP
  - Feasible region is empty, no point satisfies all the constraints
- Unbounded LP
  - Probably, an error in the formulation

# Role of LP in Computer Science

- Mostly used in deriving approximation algorithms for NP-Hard problems
- Can be used to establish confidence that a problem is solvable (i.e., not NP-Hard)
- Can be used to solve a wide variety of factory, manufacturing, scheduling, and logistics problems without having to develop an efficient algorithm
- The main challenge is to formulate the LP model

# Next Steps

- Solving an LP with software
  - Get the LINDO package from the Internet
  - Or, other LP Solvers
  - Work on a homework for LP
- Duality
- Integer Programming (IP)
- Relation between LP and IP

# Finding Dual of an LP

- Primal

$$\max z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

s.t. $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$

$$x_j \ge 0,\ j = 1,2,\ldots,n$$

or,

$$A_{ij}x_j \le b_i,$$

$$i = 1,2,\ldots m;\ j = 1,2,\ldots,n$$

- Dual

$$\min w = b_1y_1 + b_2y_2 + \cdots + b_my_m$$

s.t. $a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1$

$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \le c_2$$

$$\vdots$$

$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \le c_n$$

$$y_i \ge 0, i = 1,2,\ldots,m$$

$$A_{ji}y_i \ge c_j,$$

$$i = 1,2,\ldots m;\ j = 1,2,\ldots,n$$

# An Example

- Giapetto Problem

$$\max z = 3x_1 + 2x_2$$

s.t.

$$x_1 \le 40,$$
$$2x_1 + x_2 \le 100,$$
$$x_1 + x_2 \le 80,$$
$$x_j \ge 0,\ j = 1,2.$$

- Dual

$$\min w = 40y_1 + 80y_2 + 100y_3$$

s.t.

$$2y_1 + y_2 + y_3 \ge 3,$$
$$y_2 + y_3 \ge 2,$$
$$y_i \ge 0, i = 1,2,3.$$

- Sometimes, an LP may need to be converted to a normal form, i.e. converting constraints with "≥" and "=" to a constraint with "≤" before finding its dual

# Interpretation of a Dual

- Provide key insight
- For our Giapetto example, in the dual
  - What are we minimizing?
  - What do the constraints mean?
- In general, finding an interpretation for the dual can shed new light on the original problem and its optimal value

# Weak Duality Theorem

- **<u>Theorem:</u>** Let $\bar{x}$ be a feasible solution to the primal (*max* problem) and $\bar{y}$ be a feasible solution to the dual. Then

$$z \text{ value for } \bar{x} \leq w \text{ value for } \bar{y}$$

- **<u>Proof:</u>**
- What other results can you derive from this Theorem?

# Additional Results on Duality

- If the primal is unbounded, then the dual is infeasible,
- If the dual is unbounded, then the primal is infeasible,
- If equality holds for some $\bar{x}^*$ and $\bar{y}^*$, i.e.,

$$z \text{ value for } \bar{x}^* = w \text{ value for } \bar{y}^*$$

  - then, $\bar{x}^*$ is optimal for the primal and $\bar{y}^*$ is optimal for dual.

# Integer Programming (IP)

- If all decision variables in an LP can take on only integral values, it becomes an IP
- Most problems we encounter in networking, are IP, and not LP; Why?

# Relaxing an IP to an LP

- IP

$$\max z = 21x_1 + 11x_2$$

s.t. $7x_1 + 4x_2 \le 13,$

$$x_j \ge 0, x_j \in Z, j = 1,2.$$

- LP Relaxation

$$\max z = 21x_1 + 11x_2$$

s.t. $7x_1 + 4x_2 \le 13,$

$$x_j \ge 0, j = 1,2.$$

- Is a feasible solution to the IP also a feasible solution to its LP relaxation?
- Vice versa?
- Is an optimal solution to the IP an optimal solution for its LP relaxation?
- Vice versa?

# Can We Use LP to Solve IP?

- Can we relax an IP into an LP, solve the LP, and round off the optimal solution to obtain an optimal solution to the original IP?
  - or close to optimal solution?
- Consider the preceding IP problem
  - Sometimes, the optimal solution to an LP relaxation *may not be even feasible* for the original IP problem
- Is there a relation between the optimal solution to an IP and that of its relaxed version?

# A Key Result

- The following relation holds for all Primal (P) IP's (*max* problem), their relaxed LP versions, and their corresponding duals (D)

$$OPT_P^{IP} \leq OPT_D^{IP} \leq OPT_D^{LP} = OPT_P^{LP}$$

- How does this relation change for a *min* problem?
- Can this result be used to derive approximation guarantees? How?